Math 677.
Lecture 3.

We know: $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ has unique sol.

$$x(t) = e^{At} x_0$$

It remains to compute e^{At} for all A.

Case I. A has real distinct $\lambda_1, \dots, \lambda_n$

$$\Rightarrow e^{At} = P e^{\Lambda} P^{-1} \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$e^{\Lambda} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} \quad P = \{v_1, \dots v_n\}$$

Case II. A has complex eigenvalues

$$\lambda_j = a_j \pm i b_j$$

From linear algebra:

Thm 1: If $A \in \mathbb{R}^{2n \times 2n}$ with eigenvalues

$$\lambda_j = a_j \pm i b_j, \quad j = 1, \dots n$$

Let $w_j = u_j \pm i v_j$ be the corresp. eigenvectors

Then $P = [v_1 u_1 \dots v_n u_n]$ is invertible

and $P^{-1} A P = \operatorname{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$

$$= \begin{pmatrix} \begin{array}{cc|cc} a_1 & -b_1 & & \\ b_1 & a_1 & & \\ \hline & & a_2 & -b_2 \\ & & b_2 & a_2 \end{array} & \\ & \ddots \end{pmatrix}$$

$\Rightarrow$ Corollary:

If $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = P\,\mathrm{diag}\left\{ e^{a_j t} \begin{bmatrix} \cos\beta_j t & -\sin\beta_j t \\ \sin\beta_j t & \cos\beta_j t \end{bmatrix} \right\} P^{-1} x_0$

$$= P \left[\begin{array}{cc|c} e^{a_1 t}\cos\beta_1 t & -e^{a_1 t}\sin\beta_1 t & \\ e^{a_1 t}\sin\beta_1 t & e^{a_1 t}\cos\beta_1 t & O \\ \hline & & e^{a_2 t}\cos\beta_2 t \;\; \cdot \\ & O & \cdot \;\; \cdot \end{array} \right] \times P^{-1} x_0$$

In case when $\lambda_1, \dots \lambda_k$ are real
$\lambda_{k+1}, \dots \lambda_n$ are complex

$$e^{At} \text{ looks like } \left[\begin{array}{cc|c} e^{\lambda_1 t} & 0 & \\ 0 & e^{\lambda_k t} & O \\ \hline & O & \mathrm{diag}\left[e^{a_j t} \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \right] \end{array} \right]$$

Ex. $A = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & -4 & 5 \\ 0 & -5 & 2 \end{array} \right)$ $\qquad \lambda_1 = 2$

$$\begin{vmatrix} -4-\lambda & 5 \\ -5 & 2-\lambda \end{vmatrix} = -8 + 4\lambda - 2\lambda + \lambda^2 + 25 = \lambda^2 + 2\lambda + 17$$

$$\lambda_{23} = \frac{-2 \pm \sqrt{4 - 68}}{2} = -1 \pm 4i$$

$a \pm i\beta$

$a = -1$
$\beta = 4$

v_1: $(A-2I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 5 \\ 0 & -5 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\downarrow x_2 = 0 \quad x_3 = 0$

v_2: $\begin{pmatrix} -4-(-1+4i) & 5 \\ -5 & 2-(-1+4i) \end{pmatrix} = \begin{pmatrix} -3-4i & 5 \\ -5 & 3-4i \end{pmatrix} \times(-3+4i)$

$= \begin{pmatrix} 25 & 5(-3+4i) \\ -5 & 3-4i \end{pmatrix} = \begin{pmatrix} 25 & -5(3-4i) \\ -5 & 3-4i \end{pmatrix}$

$(a+bi)(a-bi) = a^2+b^2 \qquad \Rightarrow (-5)v_2^{(1)} + (3-4i)v_2^{(2)} = 0$

$$v_2 = \begin{pmatrix} 3-4i \\ 5 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}_{u} + \underbrace{\begin{pmatrix} -4 \\ 0 \end{pmatrix}}_{v} i$$

$$v_3 = u - vi = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix} i$$

$$P = \cancel{\begin{pmatrix} 1 & 0 & 0 \\ 0 & +3 & -4 \\ 0 & 5 & +0 \end{pmatrix}} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t}\cos 4t & -e^{-t}\sin 4t \\ 0 & e^{-t}\sin 4t & e^{-t}\cos 4t \end{pmatrix} P^{-1} x_0$$

Special case:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \Rightarrow e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

Case III. (§1.7) Repeated eigenvalues.

Ex. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\lambda_1 = \lambda_2 = 1$ $\longrightarrow$ (a) $\{v_1, v_2\}$ - form a basis

(b) v_1 is the only lin. indep. eig. vector

In case III(a) $A = P \Lambda P^{-1}$

with $P = [v_1 \dots v_n]$

$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

In case III(b), we need <u>generalized eigenvectors</u> in order to get a basis in $\mathbb{R}^n$.

Def. If $Ax = \lambda x$ (x is an eigenvector)

λ - eigenvalue of multiplicity $k \le n$

then there is a vector w s.t.

$(A - \lambda I)^m w = 0$ for $m \le k$

$\Rightarrow$ w is called a generalized eigenvector of order m.

Alternatively, can find w by solving

$(A - \lambda I) w_1 = x$ wher x is the eigenvector

$(A - \lambda I) w_2 = w_1$

$\dots$

Def. If A is s.t. $A^k = 0$, but $A^{k-1} \ne 0$

$\Rightarrow$ A is nilpotent of order k.

From linear algebra:

<u>Thm 2.</u> $A \in \mathbb{R}^{n\times n}$ $\lambda_1, \dots \lambda_n$ - real eigenvalues

Then there is a basis of generalized eigenvectors $w_1, \dots w_n$.

Moreover, $P = [w_1, \dots w_n]$ is invertible and $A = S + N$, where $S = P\,\mathrm{diag}(\lambda_j) P^{-1}$

N - nilpotent & $SN = NS$.

of order $k \le n$

<u>Corollary</u>: $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ A as above $\Rightarrow$

$$x(t) = P\,\mathrm{diag}(e^{\lambda_j t}) P^{-1} \left[I + Nt + \frac{(Nt)^2}{2!} + \dots \frac{N^{k-1} t^{k-1}}{(k-1)!} \right] x_0$$

<u>Special cases</u>:

1) if $\lambda_1, \dots \lambda_n$ are distinct $\Rightarrow S = A,\ N = 0$ $P = \underbrace{[v_1, \dots v_n]}_{\text{eig. vectors}} \Rightarrow$ old formula

2) $\lambda = \lambda_1 = \dots = \lambda_n \Rightarrow \underbrace{P = I}_{\text{gen. eigenvectors}} \Rightarrow x(t) = \mathrm{diag}(e^{\lambda t}) \cdot (I + Nt + \dots) x_0$

3) $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \Rightarrow e^A = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{S} + \underbrace{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}_{N}$$

Ex. $\begin{pmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ $\quad \det(A-\lambda I) = (\lambda+1)^2(\lambda+2)$

$$\lambda_1 = -2 \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Need generalized eig. vector v_3:

$$(A+I)^2 v_3 = 0 \quad (\text{or } (A+I)v_3 = v_2)$$

$$(A+I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = v_2 + \underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

pick $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S = P \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$N = A - S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad N^2 = 0 \quad k = 2$$

nilpotent

$$x(t) = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{P} \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}}_{P^{-1}} (I + Nt)\, x_0$$