Periodic orbits, limit cycles.

$\dot{x} = f(x)$

Def. A <u>cycle</u> is any closed solution curve Γ which is not an equilibrium

→ <u>stable</u> if $\forall \varepsilon > 0\ \exists U_\delta(\Gamma)$ s.t.
$\forall x \in U \quad d(\Gamma_x^+, \Gamma) < \varepsilon$

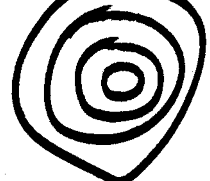
$\forall x \in U \quad t \geq 0 \quad d(\varphi(t,x), \Gamma) < \varepsilon$

→ <u>unstable</u> if it's not stable

→ <u>asymp. stable</u> if it is stable and
$\forall x \in U \quad \lim_{t \to \infty} d(\varphi(t,x), \Gamma) = 0.$

Periodic orbit: $\varphi(t+T, x) = \varphi(t,x)$

Center for linear system:



$T = \text{const}$

In general, $T \nearrow$ as you go away from equilibrium.

$\Gamma: x = \gamma(t), \ 0 \leq t \leq T$

asymp. stable $\Leftarrow \int_0^T \nabla \cdot f(\gamma(t))\,dt \leq 0$

In 2D, asymp. stable $\Leftrightarrow \int_0^T \nabla \cdot f(\gamma(t))\,dt < 0$

ω-limit cycle: asymp. stable cycle.
is an attractor

Local invariant manifolds:

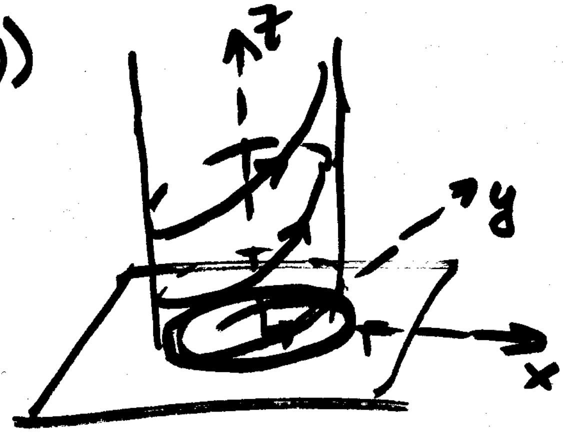
$$S = \{x \in N \mid d(\varphi_t(x), \Gamma) \to 0 \text{ as } t \to \infty, \ \varphi_t(x) \in N, \ t \geq 0\}$$
stable

$$U = \{x \in N \mid d(\varphi_t(x), \Gamma) \to 0 \text{ as } t \to -\infty, \ \varphi_t(x) \in N, \ t \leq 0\}$$
unstable

Global manifolds:

$$W^s(\Gamma) = \bigcup_{t \leq 0} \varphi_t(S(\Gamma))$$

$$W^u(\Gamma) = \bigcup_{t \geq 0} \varphi_t(U(\Gamma))$$



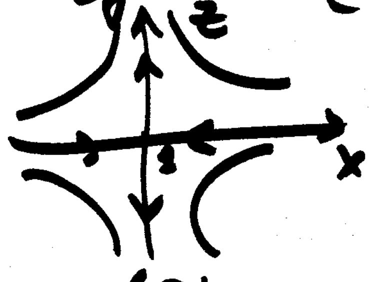
Ex.
$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \\ \dot{z} = z \end{cases}$$

Invariant sets: $\{z\text{-axis}\} \cup \{x^2+y^2=1\} \cup \{xy\text{-plane}\}$, $\{xy\text{-plane}\} = W^s(\Gamma)$

Γ: $x(t) = (\cos t, \sin t, 0)$

$W^u(\Gamma) = \{x^2+y^2=1\}$

cylinder $\{x^2+y^2=1\}$ - attracting set



periodic orbit of saddle type

In 2D (Planar system):

Def. Limit cycle in 2D is a cycle that is either an ω-limit set or α-limit set for some trajectory other than Γ.

ω-limit cycle (α) is stable (unstable) if it is an ω-limit (α) for all traj. in some $N_\delta(\Gamma)$.

semi-stable if it's an ω-limit for some traj. and an α-limit for some others.

<u>Ex.</u>
$$\begin{cases} \dot{x} = -y + x(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} \\ \dot{y} = x + y(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} \end{cases} \quad \text{for } x^2+y^2>0$$

$\dot{x}=\dot{y}=0$, $x^2+y^2=0$.

$$\begin{cases} \dot{r} = r^3 \sin\frac{1}{r} \\ \dot{\theta} = 1 \end{cases} \qquad \Gamma_n : r = \frac{1}{\pi n}, n \in \mathbb{Z}$$

accumulating at $(0,0)$.

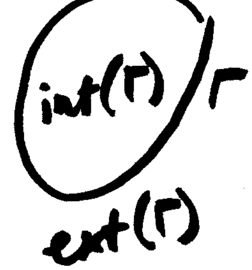
Γ_{2n} – stable

Γ_{2n+1} – unstable

<u>Thm.</u> If a traj. in $\mathrm{ext}(\Gamma)$ has Γ as a

limit cycle

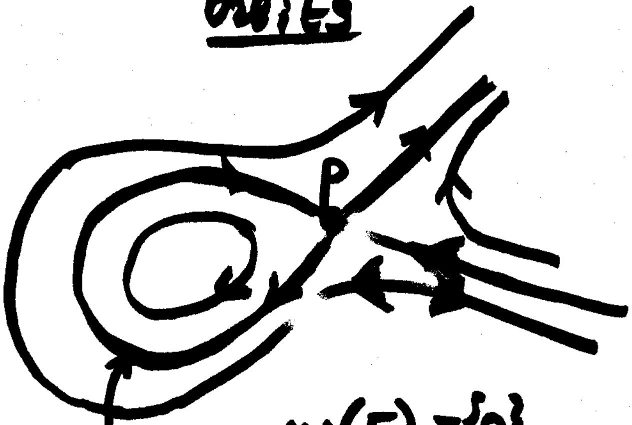
ω-limit, then all traj. in $N_\delta(\Gamma) \subset \mathrm{ext}(\Gamma)$ have Γ as a ω-limit.



~~If~~ $\varphi(t,x) \subset \mathrm{ext}(\Gamma)$ spirals toward Γ, all of them spiral toward Γ, ~~and~~ i.e. intersect a straight line $\perp \Gamma$ ~~the~~ inf. many times at $t = t_n$, $t_n \to \infty$.

<u>Thm.</u> (Dulac). For any bdd region in $\mathbb{R}^2$, any analytic system has a fin. number of limit cycles.

Homoclinic orbits



Γ

$\omega(\Gamma) = \{p\}$
$\alpha(\Gamma) = \{p\}$

$p = q$

Heteroclinic orbits

p, q - saddles

$\omega(\Gamma_1) = q \quad \omega(\Gamma_2) = p$
$\alpha(\Gamma_1) = p \quad \alpha(\Gamma_2) = q$

$p \neq q$

$$\begin{cases} \dot{x} = y \\ \dot{y} = x + x^2 \end{cases} \quad (1)$$

Hamiltonian

$$H = \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^3}{3}$$

$$y^2 - x^2 - \frac{2}{3}x^3 = c$$

$\Gamma : \{c = 0\} \subset W^s(0) \cup W^u(0)$

Undamped pendulum

$$\ddot{x} + \sin x = 0. \quad (2)$$

Newtonian

__Homoclinic__ orbit : contained in both $W^s(p), W^u(q)$
p, q - saddles, $p = q$

__Heteroclinic__ orbit : same but $p \neq q$.

Flow on $\Gamma \cup \{0\}$ is separatrix cycle in (1)
Flow on $\Gamma_1 \cup \Gamma_2 \cup \{p\} \cup \{q\}$ - finite separatrix cycle in (2)

__Compound__ separatrix cycles : union of compatibly oriented separatrix cycles.