# Math 677.
# Lecture 23

$\dot{x} = f(x)$, $f \in C^1(E)$, $E$-open $\subset \mathbb{R}^n$

$\varphi(\cdot, x)$ - solution curve going through $x \in E$

$\Gamma_{x_0} = \{x \in E \mid x = \varphi(t, x_0),\ t \in \mathbb{R}\}$ - solution moves along this trajectory starting at $x_0 \in E$.

**Def.** $p \in E$ $\exists t_n \to \infty$ s.t. $\lim_{n\to\infty} \varphi(t_n, x) = p$

$\Rightarrow p$ is an <u>$\omega$-limit</u> pt. of trajectory $\varphi(\cdot, x)$.

$p \in E$ $\exists t_n \to -\infty$ s.t. $\lim_{n\to\infty} \varphi(t_n, x) = p$

$\Rightarrow p$ is a <u>$\alpha$-limit</u> pt. of traj. $\varphi(\cdot, x)$

$\omega(\Gamma)$ = set of all $\omega$-limit pts of $\Gamma$

$\alpha(\Gamma)$ = set of all $\alpha$-limit pts of $\Gamma$.

$\omega(\Gamma) \cup \alpha(\Gamma)$ = limit set of $\Gamma$.

**Thm.** $\omega(\Gamma), \alpha(\Gamma)$ - closed subsets of $E$

If $\Gamma \subset K$-compact set in $\mathbb{R}^n \Rightarrow \alpha(\Gamma), \omega(\Gamma)$ are non-empty, compact connected subsets of $E$.

**Proof.**

1) To show: $\omega(\Gamma)$ - closed subset of $E$.

$p_n$ - sequence of pts in $\omega(\Gamma)$ s.t. $p_n \to p \in \mathbb{R}^n$

To prove: $p \in \omega(\Gamma)$.

$x_0 \in \Gamma$ then since $p_n \in \omega(\Gamma)$

$\exists t_k^{(n)} \to \infty$ s.t. $\lim_{k\to\infty} \varphi(t_k^{(n)}, x_0) = p_n$

Take $t_k^{(n+1)} > t_k^{(n)}$ subsequence

$\Rightarrow \forall n \ge 2 \ \exists K(n) > K(n-1)$ s.t. $\forall k \ge K(n)$

$|\varphi(t_k^{(n)}, x_0) - p_n| < \frac{1}{n}$

Let $t_n = t_{K(n)}^{(n)}$, $t_n \to \infty$ and

$$|\varphi(t_n, x_0) - p| \le \underbrace{|\varphi(t_n, x_0) - p_n|}_{< \frac{1}{n}} + |p_n - p| \le$$

$$\le \frac{1}{n} + |p_n - p| \xrightarrow[n \to \infty]{} 0 \Rightarrow p \in \omega(\Gamma).$$

2) Suppose $\Gamma \subset K$ (compact subset of $\mathbb{R}^n$)

$$\underbrace{\varphi(t_n, x_0)}_{\in \Gamma \subset K} \to p \in \omega(\Gamma) \Rightarrow p \in K$$

so $\omega(\Gamma)$ has to be compact since it's a closed subset of a compact set.

3) $\varphi(n, x_0) \in K$ contain a convergent subsequence converging to a pt in $\omega(\Gamma) \subset K$.

$\Rightarrow \omega(\Gamma) \neq \emptyset$.

4) Suppose $\omega(\Gamma) = \underbrace{A \cup B}_{\text{disjoint, closed nonempty}}$

(Remark: Boundedness is essential since

← example of a disconnected $\omega(\Gamma)$ ).

$$d(A,B) = \inf_{\substack{x \in A \\ y \in B}} |x - y| = \delta$$

$\exists t$ s.t. $d(\varphi(t, x_0), A) > \frac{\delta}{2}$ since $A \subset \omega(\Gamma)$.

$\exists t$ s.t. $d(\varphi(t, x_0), A) < \frac{\delta}{2}$

$d(\varphi(t,x_0),A)$ - continuous fct of time.

$\exists t_n \to \infty$ s.t. $d(\varphi(t_n,x_0),A) = \frac{\delta}{2}$

But then ~~dp~~ $\underset{K}{\varphi(t_n,x_0)}$ has a convergent

subsequence with limit $p \in \omega(\Gamma)$, s.t.

$d(p,A) = \frac{\delta}{2}$

Then $d(p,B) \geq d(A,B) - d(p,A) = \frac{\delta}{2}$

$\Rightarrow \underset{p \notin A}{p \notin B}, \Rightarrow p \notin \omega(\Gamma) \Rightarrow$ contradiction.

$\Rightarrow \omega(\Gamma)$ is connected.

$\alpha(\Gamma)$ is also connected by a similar argument.

<u>Thm 2</u>. $p \in \omega(\Gamma) \Rightarrow \Gamma_p \subset \omega(\Gamma)$.

$p \in \alpha(\Gamma) \Rightarrow \Gamma_p \subset \alpha(\Gamma)$

$\Rightarrow \underline{\varphi_t(p) \subset \omega(\Gamma)}$, ($\omega, \alpha$ - sets are invariant wrt $\varphi(\cdot, x)$)

<u>Pf</u>. $p \in \omega(\Gamma)$, $\Gamma$ - traj. of $\varphi(\cdot, x_0)$

Take $q = \varphi(\tilde{t}, p)$

Since $p \in \omega(\Gamma)$ $\exists t_n \to \infty$ s.t. $\varphi(t_n, x_0) \xrightarrow[n\to\infty]{} p$

By continuity wrt initial conditions,

$$\varphi(\underset{\downarrow\,\infty}{t_n + \tilde{t}}, x_0) = \varphi(\tilde{t}, \underset{\downarrow\, p}{\varphi(t_n, x_0)}) \to \varphi(\tilde{t}, p) = q$$

$\Rightarrow q \in \omega(\Gamma)$.

$\Rightarrow \begin{matrix}\omega(\Gamma) \\ \alpha(\Gamma)\end{matrix}$ - closed invariant subsets of $R^n$

**Def.** $A$ - <u>Attracting set</u> is a closed invariant subset of $E \subset \mathbb{R}^n$ s.t.

$\exists\, U$ - nbhd of $A$ s.t. $\forall x \in U$, $\varphi_t(x) \in U$

$\forall t \geq 0$ $\varphi_t(x) \underset{t\to\infty}{\longrightarrow} A$

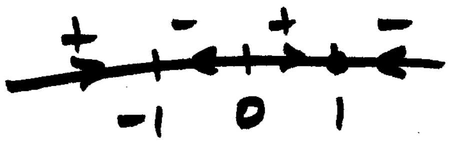
<u>Attractor</u> is an attracting set that contains a dense orbit.

<u>Ex.</u> $\begin{cases} \dot{x} = -y + \mu x(1-x^2-y^2) \\ \dot{y} = x + \mu x(1-x^2-y^2) \end{cases}$ $\mu > 0$

$\begin{cases} \dot{\theta} = 1 \\ \dot{r} = \mu r(1-r^2) \end{cases}$

$\dot{r} > 0$, $0 < r < 1$

$\dot{r} < 0$, $r > 1$




$\Rightarrow r = 1 : S = (x^2+y^2=1) = \omega(\Gamma_p)$ is a <u>limit cycle</u>

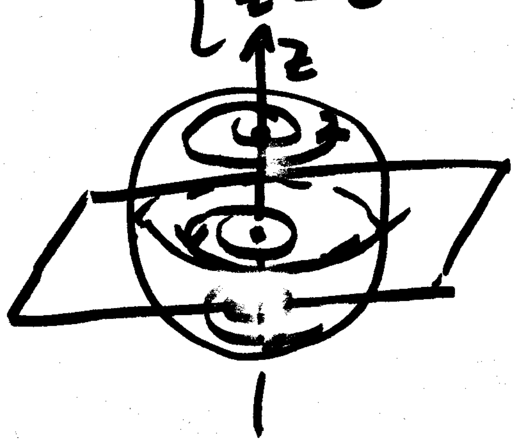
attracting set for any $\Gamma_p$, $p \neq (0,0)$.

<u>Ex.</u> $\begin{cases} \dot{x} = -y + x(1-x^2-y^2-z^2) \\ \dot{y} = x + y(1-x^2-y^2-z^2) \\ \dot{z} = 0 \end{cases}$

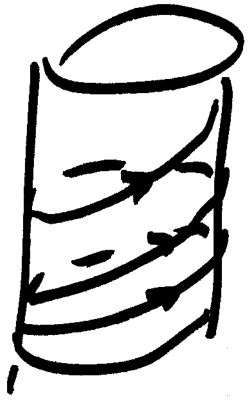
$\omega(\Gamma_p) = S^2 \cup \{x=y=0, z \geq 0\}$

$\{z = z_0\}$ - invariant set

$|z_0| < 1 \Rightarrow$ all $\omega$-limit sets are limit cycles



**Ex.** $\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \\ \dot{z} = \alpha \end{cases}$



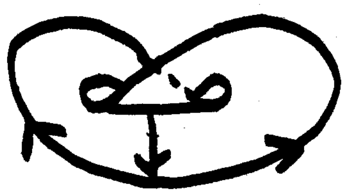
Invariant sets: $\{z\text{-axis}\} \cup S^1$

**Ex.** $\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$

Lorenz system

$\sigma = 10, \ \rho = 28, \ \beta = \frac{8}{3}$

strange attractor



← Branched surface

↳ has countable number of periodic orbits

uncountable set of non-periodic motions

has a dense orbit.