Math 677.
Lecture 22.

Global Theory

Def. A dynamical system on E is a map $\varphi: R \times E \to E$, $\underline{\varphi \in C^1}$
s.t. $\varphi_t(x) = \varphi(t, x)$ satisfies (i) $\varphi_0(x) = x$, $\forall x \in E$
(ii) $\varphi_t \circ \varphi_s(x) = \varphi_{t+s}(x)$
$\forall s, t \in R$, $x \in E$

Ex. $\varphi(t, x) = e^{At} x$
- solution to IVP $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$

→ $\varphi(t, x)$ - dyn. system on $E \subset \mathbb{R}^n$ $\Rightarrow$ $f(x) = \frac{d}{dt}\varphi(t, x)\Big|_{t=0}$
defines a C^1-vector field on E, s.t.
$\forall x_0 \in E$ $\varphi(t, x_0)$ - solution to IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$

→ $\forall x_0 \in E \Rightarrow$ max. interval of existence for $\varphi(t, x_0)$ is $I(x_0) = (-\infty, \infty)$.

→ if $\dot{x} = f(x)$, $f \in C^1(E)$, E - open set in $\mathbb{R}^n$
$\varphi(t, x_0)$ - sol. to IVP, $x_0 \in E$
then $\varphi(t, x)$ - dyn. system $\Leftrightarrow$ $I(x_0) = R$

Def. $f, g \in C^1(E_2)$, E_1, E_2 - open $\subset \mathbb{R}^n$
$\hat{C}^1(E_1)$ $\left.\begin{array}{l} \dot{x} = f(x) \\ \dot{x} = g(x) \end{array}\right\}$ top. equivalent if

$\exists H$ - homeo: $E_1 \xrightarrow{\text{onto}} E_2$ which maps traj. of one system onto traj. of the other, and preserves orientation by time.

If in addition H preserves parameterization by time, the systems are top. Conjugate.

Observation:

(1) $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \underset{\text{top. equivalent}}{\sim} \begin{cases} \dot{x} = \dfrac{f(x)}{1+|f(x)|} \quad (*) \\ x(0) = x_0 \end{cases}$

Time rescaling: $\tau(t) = \int_0^t (1+|f(x(s))|)ds$

in τ, $\dot{x}(\tau) = \frac{dx}{dt} \cdot \frac{dt}{d\tau} = \frac{dx}{dt} / \frac{d\tau}{dt} \Rightarrow$

$H = Id$ (same portrait) $\dot{x}(\tau) = \dfrac{f(x)}{1+|f(x)|}$

$x(t(\tau))$ is solution to $(*)$, with $t(\tau)$ - inverse of $\tau(t)$ given above

$\tau(t)$ strictly increasing.

$\tau(t) : I_0(x(t)) \longrightarrow I_0(x(t(\tau)))$

$(\alpha, \beta) \longrightarrow (-\infty, \infty)$

Can always do this as long as $f \in C'(\mathbb{R}^n)$

Ex.1 $\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases}$ $\quad \exists! \, x(t) = \dfrac{x_0}{1 - x_0 t}$

$I(x_0) = \begin{cases} (-\infty, \frac{1}{x_0}), & x_0 > 0 \\ (\frac{1}{x_0}, \infty), & x_0 < 0 \\ (-\infty, \infty), & x_0 = 0 \end{cases}$

Consider $\begin{cases} \dot{x} = \dfrac{f(x)}{1+|f(x)|} = \dfrac{x^2}{1+x^2} \\ x(0) = x_0 \end{cases}$

$$\exists!\text{ sol. } x(t)=\begin{cases}\frac{1}{2}\left(t+x_0-\frac{1}{x_0}+\frac{x_0}{|x_0|}\sqrt{t^2+2(x_0-\frac{1}{x_0})t+(x_0+\frac{1}{x_0})^2}\right. & x_0\neq 0\\ 0, & x_0=0\end{cases}$$

$I(x_0)=(-\infty,\infty)$, if $x_0>0$ $\quad x(t)\xrightarrow[t\to\infty]{}\infty$, $x(t)\xrightarrow[t\to-\infty]{}0$

$x_0<0$ $\quad x(t)\xrightarrow[t\to\infty]{}0$ $\quad x(t)\xrightarrow[t\to-\infty]{}-\infty$

$$\tau(t)=t+\frac{x_0^2 t}{1-x_0 t}\doteq t,\ x_0=0$$

maps $I(x_0)$ onto $(-\infty,\infty)$

$\uparrow$ previous system

Ex. 2. $\begin{cases}\dot{x}=\frac{+1}{2x}\\ x(0)=x_0\end{cases}$ $\quad \exists!$ sol. $x=\sqrt{t+x_0^2}$

$I(x_0)=(-x_0^2,\infty)$

$f(x)=\frac{1}{2x}\in C^1(0,\infty)$

If we take $\begin{cases}\dot{x}=\dfrac{+\frac{1}{2x}}{1+\frac{1}{2x}}=\dfrac{1}{2x+1}\\ x(0)=x_0\end{cases}$ $\exists!$ sol.

$$x=-\frac{1}{2}+\sqrt{t+(x_0+\frac{1}{2})^2}$$

$$\mathbb{R}\neq I(x_0)=\left(-(x_0+\tfrac{1}{2})^2,\infty\right)$$

the trick does not work since $E=(0,\infty)\neq\mathbb{R}$

Thm 1. $f\in C^1(\mathbb{R}^n)\Rightarrow$ system (1) is top. equivalent to system ⊛. In other words, we can always talk about dyn. system when $f\in C^1$ on entire plane.

(you can always rescale time)

Sketch of proof:

1) $f \in C^1(\mathbb{R}^n) \Rightarrow \dfrac{f(x)}{1+|f(x)|} \in C^1(\mathbb{R}^n)$

Define $x(t)$ - solution to IVP $\begin{cases} \dot{x} = \dfrac{f}{1+|f|} \\ x(0) = x_0 \end{cases}$

on $I(x_0) = (\alpha, \beta)$

$$\Rightarrow x(t) = x_0 + \int_0^t \frac{f(x(s))}{1+|f(x(s))|} ds \quad \forall t \in (\alpha, \beta)$$

$$\Rightarrow |x(t)| \le |x_0| + \int_0^{|t|} ds = |x_0| + |t| \;, \; \forall t \in (\alpha, \beta)$$

2) $\Rightarrow$ solutions of IVP are contained in the compact set in $\mathbb{R}^n$: $\{x \in \mathbb{R}^n \mid |x(t)| \le |x_0| + \beta\}$ if $\beta < \infty$

then $\beta = \infty \Rightarrow I(x_0) = (\alpha, \infty)$

3) Similarly $\alpha = -\infty \Rightarrow I(x_0) = (-\infty, \infty)$.

Thm 2. $f \in C^1(E)$, E - open $\subset \mathbb{R}^n$
$\Rightarrow \exists F \in C^1(E)$ s.t. $\dot{x} = F(x)$ dyn. system on E, top. equivalent to $\dot{x} = f(x)$ on E.

Thm 3. $f \in C^1(\mathbb{R}^n)$, global Lipschitz continuity
$|f(x) - f(y)| \le M|x-y|$, $\forall x,y \in \mathbb{R}^n$
$\Rightarrow \forall x_0 \in \mathbb{R}^n$ IVP (1) $\exists!$ sol. defined $\forall t \in \mathbb{R}$.
$(I(x_0) = (-\infty, \infty))$

Thm 4. $f \in C^1(M)$, M - compact manifold
$\Rightarrow \forall x_0 \in M \; \exists!$ sol. of (1).

Global stability of $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ is associated with phenomena s.a.s cycles, ω-limits and periodic solutions.

<u>Ex.</u>

$\begin{cases} \dot{x} = f(x, \mu) \\ x(0) = x_0 \end{cases}$ $Df(x_0, \mu_0)$ has $Re(\lambda_i) = 0$ for a simple pair of eigenvalues λ_i, λ_{i+1}

Implicit function theorem: $\forall \mu \sim \mu_0$
$\exists!$ equilibrium pt $x_\mu \sim x_0$.
if eigenvalues of $Df(x_\mu, \mu)$ cross $\{Re\, x = 0\}$ at $\mu = \mu_0 \Rightarrow$ dimension of W^s, W^u will change $\Rightarrow$ structurally unstable v.f. at x_0 for the bifurcation value μ_0.

$\begin{cases} \dot{x} = -y + x(\mu - x^2 - y^2) \\ \dot{y} = x + y(\mu - x^2 - y^2) \end{cases}$ $(0,0)$ - crit. pt.

$Df(0, \mu) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$

$(0,0)$ - focus:
stable if $\mu < 0$
unstable if $\mu > 0$
focus or center if $\mu = 0$.

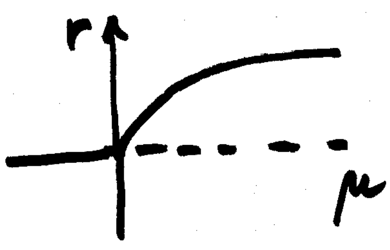
$\cancel{\mu}^2 + 1 = 0$
$(\mu - \lambda)^2 + 1 = 0$
$(\mu - \lambda)^2 = -1$
$\lambda = \mu \pm \sqrt{i}$

$\mu_0 = 0$: <u>Hopf</u> bifurcation

In polar coordinates: $\begin{cases} \dot{r} = r(\mu - r^2) \\ \dot{\theta} = 1 \end{cases}$

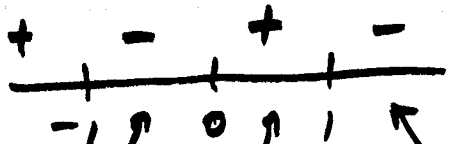
$\mu \leq 0$

$\mu > 0$
$\dot{r} = 0 \quad r^2 = \mu$
limit cycle

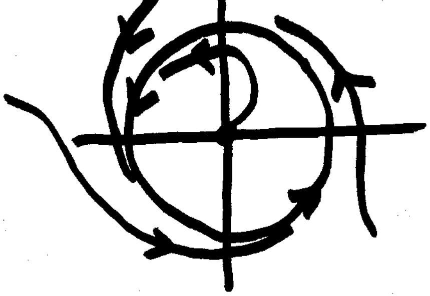


$\gamma_\mu(t) = \sqrt{\mu}(\cos t, \sin t)$
$\nwarrow$ 1-parameter family of limit cycles

$$\mu = 1 : \begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \quad \rightarrow \theta = t + \theta_0 \end{cases}$$



$\dot{r} > 0$ $\quad$ $\dot{r} < 0$ $\quad$ outward $\quad$ $\dot{r} < 0$ inward spiral



counterclockwise spiraling around the limit cycle : $\Gamma = \{ r = 1 \}$