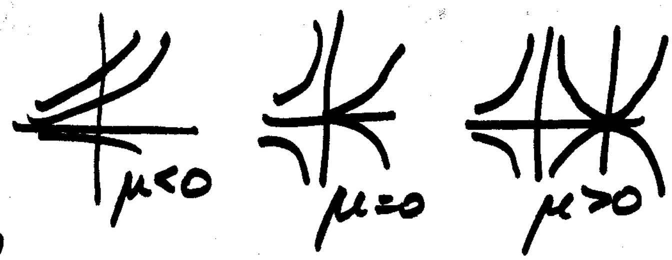
# Math 677.
# Lecture 21.

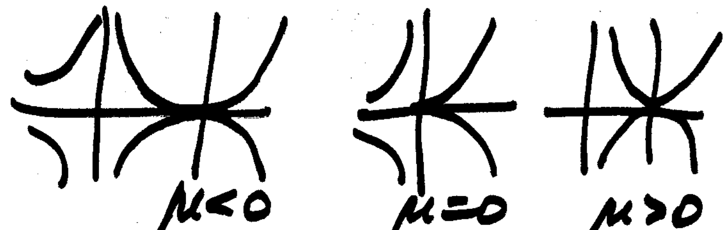
**Exam:** Tue Dec 15 4:30-7:15

2d systems, nonhyperbolic crit. pt bifurcations

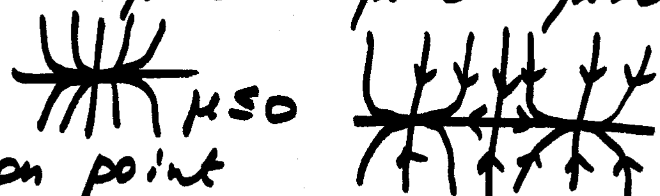
1) $\begin{cases} \dot{x} = \mu - x^2 \\ y = -y \end{cases}$ Saddle-node bifurcation




2) $\dot{x} = \mu x - x^2$ transcritical
$y = -y$




3) $\begin{cases} \dot{x} = \mu x - x^3 \\ y = -y \end{cases}$ pitchfork




All cases: $x_0 = 0 \leftarrow$ bifurcation point
$\mu_0 = 0 \leftarrow$ bifurcation value of $\mu$.

## Higher-codimension bifurcations.

(1) $\dot{x} = f(x, \mu)$ $\mu \in \mathbb{R}^m$
$f(x_0, \mu_0) = 0$, $Df(x_0, \mu_0)$ has at least one $\lambda$
$Re(\lambda) = 0$

1 Suppose that $\lambda_1 = 0$, $Re(\lambda_2, \ldots \lambda_n) \neq 0$
then $\dot{x} = F(x, \mu) \leftarrow$ flow on center manifold
close to $x_0$, $\mu \sim \mu_0$
captures the behavior of (1).

**Def.** $f(x,\mu_0) = f_0(x)$ ← structurally unstable v.f.

$f(x,\mu)$: unfolding of $f_0(x)$ at a nonhyperbolic crit. pt $x_0$

it is called a universal unfolding if any other unfolding is top. equivalent to a family of v.f. defined by (1) in a neighborhood of $x_0$.

Codimension of the bifurcation = min number of parameters in a universal unfolding.

**Some examples:**

① Saddle-node bifurcation

$F(x,0) = F_0(x) = ax^2$ normal form of the v.f. on center manifold

$\dot{x} = -x^2$ (by rescaling time)

$\quad \overset{\shortparallel}{f_0(x)}$ – structurally unstable v.f.

if $\dot{x} = -x^2 + \mu_3 x^3$ : $x = 0$, – nonhyperbolic

$x = \frac{1}{\mu_3}$ – hyperbolic

$\to \infty$ as $\mu_3 \to 0$.

$\Rightarrow$ h.o.t. will not affect the universal unfolding.

if $\dot{x} = \mu_1 + \mu_2 x - x^2$

shift $(0,0)$ to $x = \left(\frac{\mu_2}{2}, 0\right) \Longrightarrow \dot{x} = \mu - x^2$

$\mu = \mu_1 + \frac{\mu_2^2}{4}$

$\Rightarrow$ $\boxed{\dot{x} = \mu - x^2}$ universal unfolding. $\boxed{\text{Codim} = 1}$

② $\cancel{\dot{x}}$

Pitchfork $\quad F(x,0)=F_0(x)=ax^3$

$$\rightsquigarrow \dot{x} = -x^3 = f_0(x)$$

h.o.t. will not affect the phase portrait

$$\Rightarrow \dot{x} = \mu_1 + \mu_2 x + \mu_3 x^2 - x^3$$

↓ gone after translating to $(\frac{\mu_3}{3}, 0)$.

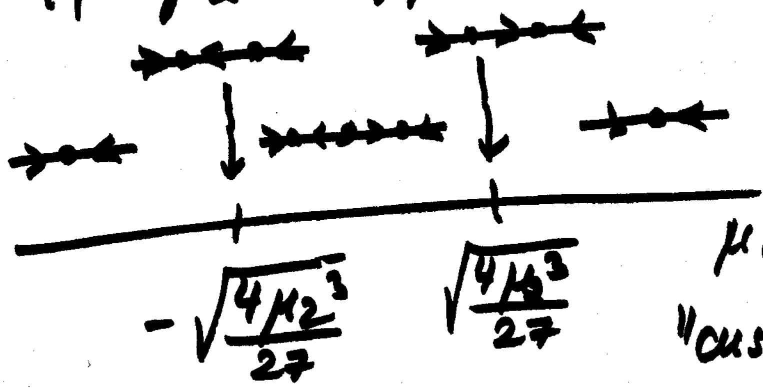
$$\Rightarrow \underbrace{\dot{x} = \mu_1 + \mu_2 x - x^3}_{\text{codim}=2} \quad \text{universal unfolding} \quad \boxed{\mu_1 + \mu_2 x - x^3 = 0}$$

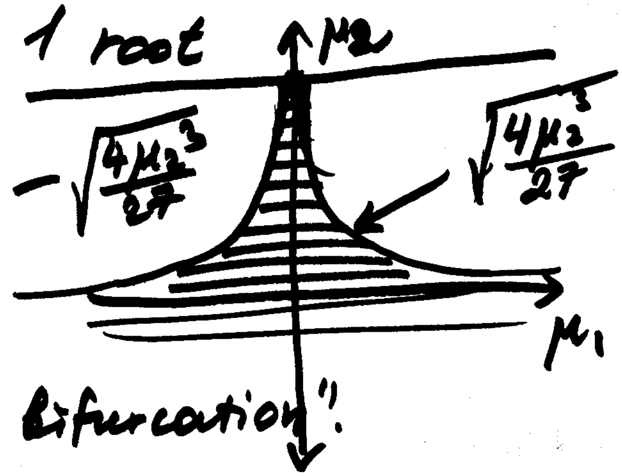
if $\mu_2 > 0$ : $\mu_1^2 < \frac{4\mu_2^3}{27} \longleftrightarrow$ 3 roots

$\mu_1^2 = \frac{4\mu_2^3}{27} \longleftrightarrow$ 2 roots

$\mu_1^2 > \frac{4\mu_2^3}{27} \longleftrightarrow$ 1 root

if $\mu_2 \le 0$, $\mu_1 \in \mathbb{R} \longrightarrow$ 1 root




"cusp bifurcation".

ex.3 in prev. lecture:

$\dot{x} = \mu x - x^3$ didn't capture cases of $\mu_1^2 = \frac{4\mu_2^3}{27} \Rightarrow$ was not a universal unfolding.