Math 677.

Lecture 19.

Hamiltonian systems.

Def. $E \in \mathbb{R}^{2n}$ - open, $H \in C^2(E)$, $H = H(x,y)$ ← energy

$$(1)\ \begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases} \quad \text{- Hamiltonian system}$$

These systems always conserve energy:

Thm 1. $\dfrac{dH}{dt} = \dfrac{\partial H}{\partial x}\cdot\dot{x} + \dfrac{\partial H}{\partial y}\cdot\dot{y} \equiv 0 \Rightarrow H(x,y) = \text{const}$ along the trajectories

Equilibria of (1) $\Leftrightarrow$ crit. points of $H(x,y)$. (NLOG translated to $(0,0)$)

Lemma. If $(0,0)$ - focus of $\begin{cases} \dot{x} = H_y(x,y) \\ \dot{y} = -H_x(x,y) \end{cases}$ (planar case)

then $(0,0)$ cannot be a strict min/max of H.

Proof: Let $(0,0)$ - stable focus

In polar coordinates, $\exists \varepsilon > 0$, $0 < r_0 < \varepsilon$, $\theta_0 \in \mathbb{R}$

s.t. $r(t, r_0, \theta_0) \to 0$ $\quad r(0) = r_0$
$|\theta(t, r_0, \theta_0)| \to \infty$ $\quad \theta(0) = \theta_0$ as $t \to +\infty$

$$H(x_0, y_0) = \lim_{t\to\infty} H(x(t, x_0, y_0), y(t, x_0, y_0)) = H(0,0)$$

it cannot be that $H(x,y) > H(0,0)$ or $H(x,y) < H(0,0)$

for all $(x,y) \in N_\varepsilon(0)\setminus\{0\}$.

Def. x_0 - crit. pt of $\dot{x} = f(x)$
x_0 - nondegenerate if $Df(x_0)$ has no zero eigenvalues.

x_0 - nondegenerate in $\mathbb{R}^2$ $\Rightarrow$ either hyperbolic pt ($Re(\lambda_i) \neq 0$)
or a center for the linearized system $\left(\begin{matrix} Re(\lambda_i) = 0 \\ Im(\lambda_i) \neq 0 \end{matrix}\right)$

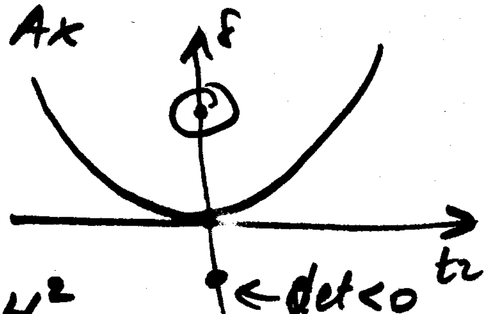
Thm 2. 1) Any nondegenerate pt of an analytic Hamiltonian system in 2D is either a saddle or a center

2) (x_0, y_0) - saddle for (1) in $\mathbb{R}^2$ $\Leftrightarrow$ saddle for H

(x_0, y_0) - local min/max of $H \Rightarrow$ center of (1) in $\mathbb{R}^2$.

Proof. $(0,0)$ - crit. pt. $H_x(0,0) = H_y(0,0) = 0$
Consider linearization $\dot{x} = Ax$

$$A = \begin{bmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{bmatrix}(0,0)$$



$tr\, A = 0 \quad \det A = H_{xx}H_{yy} - H_{xy}^2$

✓ if $\det A < 0 \Leftrightarrow$ saddle for H
$\Leftrightarrow$ saddle for (1)

if $\det A > 0$ $tr\, A = 0 \Rightarrow$ center for linearization
$\Rightarrow$ either a center ~~of~~ or focus for (1).

By Lemma 1, $(0,0)$ being a focus contradicts the statement that $\det A > 0 \Rightarrow (0,0)$ - center.

Def. $\ddot{x} = f(x)$, $f \in C^1(\alpha, \beta)$ ← Newtonian system

$$\Leftrightarrow \begin{cases} \dot{x} = y \\ \dot{y} = f(x) \end{cases} \quad (3)$$

Total energy $H(x,y) = T(y) + U(x)$

$T(y) = \frac{y^2}{2}$ - kinetic

$U(x) = -\int_{x_0}^{x} f(s)\,ds$ - potential

⇒ Newtonian ⊂ Hamiltonian

Facts:

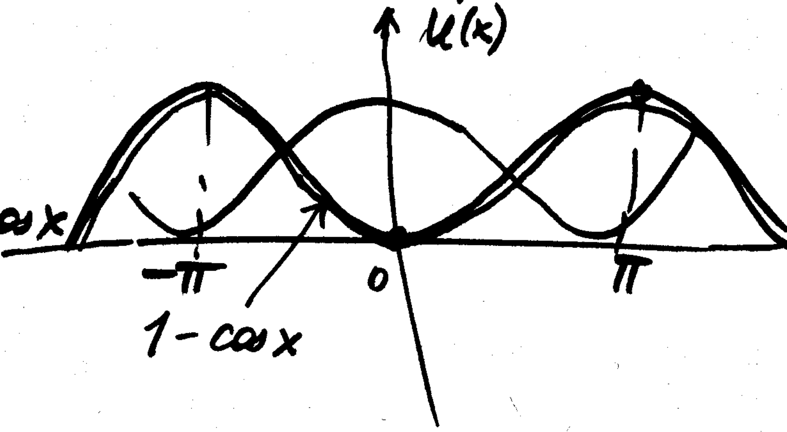
1) Crit. pts of (3) lie on x-axis
2) $(x_0, 0)$ - crit. pt. of (3) ⇔ $(x_0, 0)$ - crit. pt. of $U(x)$
3) $(x_0, 0)$ - strict local min of $U(x)$ ⇒ center for (3)
4) strict local max of $U(x)$ ⇒ saddle for (3)
5) $(x_0, 0)$ - horiz. inflection of $U(x)$ ⇒ cusp of (3)
   analytic
6) phase portrait is symmetric wrt x-axis

Ex. $\ddot{x} + \sin x = 0$ nonlinear pendulum

$\ddot{x} + x = 0$ linear pendulum

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$$

$U(x) = \int_0^x \sin s\,ds = 1 - \cos x$



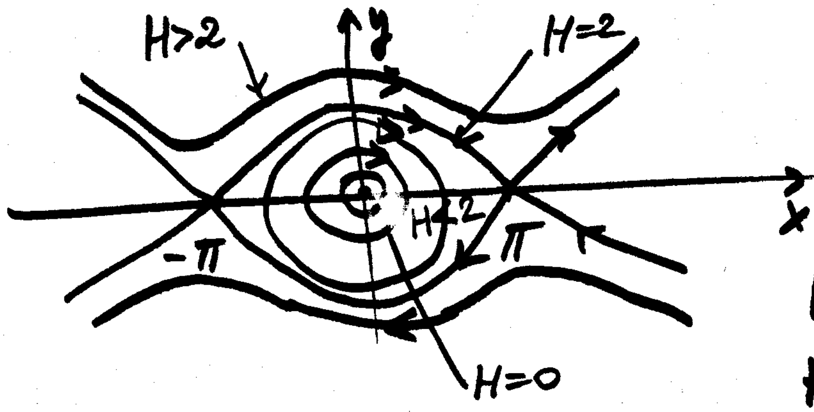
crit. pts of $U(x)$:

$x = 2\pi k$ ←

$x = \cancel{\pi k}\ \pi(1+2k)$

Relevant crit. pts: $x = 0$ ← stable (center)

$x = \pm\pi$ ← unstable (saddles)



← phase portrait

$H = \frac{y^2}{2} + 1 - \cos x$

$H(0,0) = 0$

$H(\pm\pi, 0) = 2$

Def. Gradient system:

$E \in \mathbb{R}^n$ - open, $V \in C^2(E)$

(4) $\dot{x} = -\operatorname{grad} V(x)$

Facts:

1) crit. pts of (4) $\Leftrightarrow$ crit. pts of $V(x)$

2) x_0 - regular pt (i.e. $\operatorname{grad} V(x_0) \neq 0$)
   $\operatorname{grad} V(x) \perp \{V = \text{const}\}$
   level set of V through x_0

3) x_0 - crit. pt. of V, x_0 - strict local min of V $\Rightarrow$ $V(x) - V(x_0)$ - Lyapunov function for (4) in $N_\delta(x_0)$

Thm. 1) At Any ~~non~~ regular pt of $V(x)$ ~~has~~ trajectories of (4) are perpendicular to level surfaces $V(x) = \text{const}$

2) Strict loc min of $V(x)$ are asymp. stable equilibria of (4).

Linearization of (4): $\dot{x} = Ax$

$A = \left\{ \frac{\partial^2 V}{\partial x_i \partial x_j} \right\}$ - symmetric, real λ_i, diagonalizable

<u>Thm.</u> 1) Any nondegenerate crit. pt (x_0, y_0) of (4), V-analytic $\Rightarrow (x_0, y_0)$ saddle or a node.

2) (x_0, y_0) - saddle of $V(x,y) \Rightarrow$ saddle of (4)

(x_0, y_0) strict loc min/max of $V(x,y)$ $\Rightarrow$ ~~unstab~~ stable/unstable node of (4).

<u>Def.</u> System (5) $\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$ orthogonal to $\begin{cases} \dot{x} = Q(x,y) \\ \dot{y} = -P(x,y) \end{cases}$ (6)

(i) if (5) Hamiltonian $P = H_y$, $Q = -H_x$ $\Rightarrow$ (6) Gradient system

(ii) Have same crit. pts

(iii) at x_0 - regular pt. of (5)

traj. of (5) $\perp$ traj. of (6)

Centers of (5) $\Leftrightarrow$ nodes of (6)

saddles of (5) $\Leftrightarrow$ saddles of (6)

foci of (5) $\Leftrightarrow$ foci of (6)