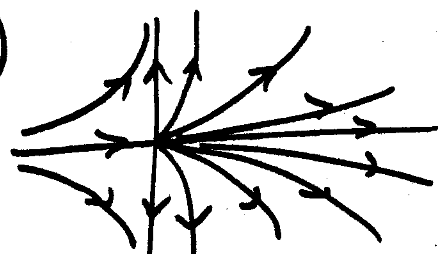
Math 677.
Lecture 18

Nonhyperbolic equilibria: nodes
foci
saddles
center

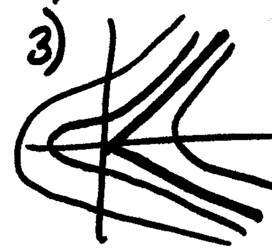
In addition, 3 more scenarios are possible:

1) 

saddle node

2) 

elliptic domain

3) 

cusp

Sectors:


hyperbolic


parabolic


elliptic

Saddle node → 1 parabolic sector in positive half-plane
2 hyperbolic sectors
3 separatrices

Elliptic domain → 1 hyperbolic sector
2 parabolic
1 elliptic
4 separatrices

Cusp → 2 separatrices
2 hyperb. sectors

Normal form theory:

$$\dot{x} = f(x) \longrightarrow \dot{x} = Jx + F(x) \text{ with "Simple" } F$$

Ex.1 $\begin{cases} \dot{x} = y + ax^2 \\ \dot{y} = x^2 + bxy + x^3 \end{cases}$ $\quad J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -ax^2 \end{pmatrix}$ — nonlinear change of variables

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -ax^2 \end{pmatrix}$$

$\Rightarrow$

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = u_1^2 + (b+2a)u_1u_2 + (1-ab)u_1^3 \end{cases}$$ cusp

$k=2 \quad b_n = b+2a, \ n=1, \ m=1 \ (k=2m)$

Case 1. $\operatorname{tr} A \neq 0$ $(\lambda_1 = 0, \lambda_2 \neq 0)$

$$\dot{x} = f(x) \quad \sim \quad \begin{cases} \dot{x} = p_2(x,y) \\ \dot{y} = y + q_2(x,y) \end{cases}$$

$p_2, q_2 \in C^\infty$, start with quadratic terms in expansion.

Claim 1: $\{0\}$ - isolated pt of $\dot{x} = f(x)$.
$y = \varphi(x)$: solution to $y + q_2(x,y) = 0$ in $N_\delta(0)$.

$$\Psi = p_2(x, \varphi(x)) = \underset{\substack{\# \\ 0}}{a_m} x^m + \dots, \quad m \geq 2$$

$\Rightarrow$ (1) m-odd, $a_m > 0$ $\Rightarrow$ $\{0\}$ - unstable node

(2) m-odd, $a_m < 0$ $\Rightarrow$ $\{0\}$ - saddle

$\Rightarrow$ (3) m-even $\Rightarrow$ $\{0\}$ - saddle-node

Case 2. $\operatorname{tr} A = 0$ $(\lambda_1 = 0, \lambda_2 = 0)$

$$\dot{x} = f(x) \quad \sim \quad \begin{cases} \dot{x} = y \\ \dot{y} = a_k x^k [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 R(x,y) \end{cases}$$

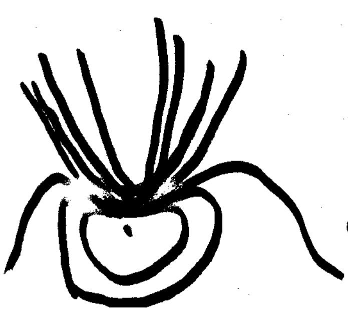
Normal form

h, g, R - analytic
$h(0) = g(0) = 0$, $k \geq 2$
$a_k \neq 0$, $n \geq 1$

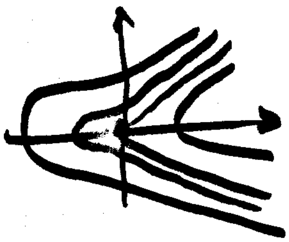
Claim 2: $k = 2m+1$, $m \geq 1$
$\lambda = b_n^2 + 4(m+1) a_k$

if $a_k > 0$ $\Rightarrow$ $\{0\}$ - saddle
if $a_k < 0$ $\Rightarrow$ (1) $b_n = 0$ $\Rightarrow$ focus or center
(2) $b_n \neq 0$, $n > m$
$n = m$, $\lambda < 0$ $\Rightarrow$ focus or center

(3) $b_n \neq 0$, n-even, $n < m$
$b_n \neq 0$, n-even, $n = m$, $\lambda \geq 0$ $\Rightarrow$ node



$\longrightarrow$ (4) $b_n \neq 0$, n-odd $n < m$
$b_n \neq 0$ n-odd $n = m$, $\lambda \geq 0$ $\Rightarrow$ elliptic domain



Claim 3: $k = 2m,\ m \geq 1$,

if (1) $b_n = 0$, $b_n \neq 0,\ n \geq m$ $\Rightarrow$ cusp

(2) $b_n \neq 0,\ n < m$ $\Rightarrow$ saddle-node

Normal form theory.

$\dot{x} = f(x) \rightsquigarrow \dot{x} = Jx + F(x)$

x_0 — nonhyperbolic pt; J — Jordan form of A

By ~~CPM~~ CMT only C-part of J matters ($\dim C < \dim A$)

reduced system to a lower-dim case

CMT — Central Manifold Theorem

Normal form theory: reduces $\dot{x} = Jx + F(x)$ to a system with a simple form of $F(x)$.

This is done by a nonlinear transformation $x = y + h(y)$, $h(y) = O(|y|^2)$, $|y| \to 0$.

This transformation is almost like identity close to $\{0\}$ $\Rightarrow$ we end up with a top. conjugate system.

Normal form calculation:

$$\dot{x} = Jx + F_2(x) + O(|x|^3) \quad |x| \to 0$$

F_2 ← 2nd deg terms

$x = y + h(y)$ by Chain Rule ⟹

$$[I + Dh(y)]\,\dot{y} = Jy + Jh(y) + F_2(y) + O(|y|^3)$$

$$[I + Dh(y)]^{-1} \cong I - Dh(y) + O(|y|^2)$$

$$\Rightarrow \dot{y} \cong (I - Dh(y) + O(|y|^2))(Jy + Jh(y) + F_2(y) + O(|y|^3))$$

$$= Jy + Jh(y) + F_2(y) - Dh(y)Jy + \ldots$$

$$= Jy + \underset{\substack{\| \\ Jh_2(y) - Dh_2(y)Jy + F_2(y)}}{\tilde{F}_2(y)} + O(|y|^3)$$

where $h = h_2(y) + O(|y|^3)$, $|y| \to 0$

Wanted: h_2 s.t. $\tilde{F}_2 = \begin{pmatrix} 0 \\ * \end{pmatrix}$ or ~~at~~ ideally $\tilde{F}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(in Ex. 1, by killing ax^2 in 1st component we introduce xy, y^2 in 2nd component).

<u>Fact</u>: $\begin{cases} \dot{x} = y + ax^2 + bxy + cy^2 + O(|x|^3) \\ \dot{y} = dx^2 + exy + fy^2 + O(|x|^3) \end{cases}$

In general, this system reduces to

$$\begin{cases} \dot{x} = y + O(|x|^3) \\ \dot{y} = dx^2 + (e + 2a)xy + O(|x|^3) \end{cases}$$

By $x \to x + h_2(x)$, $h_2 = \begin{pmatrix} a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}$

$h_2 \in H_2$ ← all 2nd deg polynomials $= \mathrm{Span}\left(\begin{pmatrix} x^2 \\ 0 \end{pmatrix} \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \begin{pmatrix} xy \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} \ldots\right.$

$$H_2 = \text{Span}\left\{\begin{pmatrix}x^2\\0\end{pmatrix}\begin{pmatrix}y^2\\0\end{pmatrix}\begin{pmatrix}xy\\0\end{pmatrix}\begin{pmatrix}0\\x^2\end{pmatrix}\begin{pmatrix}0\\y^2\end{pmatrix}\begin{pmatrix}0\\xy\end{pmatrix}\right\}$$

$$G_2 = \text{Span}\left\{\begin{pmatrix}0\\x^2\end{pmatrix}\begin{pmatrix}0\\xy\end{pmatrix}\right\}, \quad L_J(H_2) = \text{Span}\left\{\begin{pmatrix}x^2\\xy\end{pmatrix}\begin{pmatrix}xy\\y^2\end{pmatrix}\begin{pmatrix}xy\\0\end{pmatrix}\begin{pmatrix}y^2\\0\end{pmatrix}\right\}$$

$$H_2 = L_J(H_2) \oplus G_2 \qquad \text{or } L_J(H_2) = \text{Span}\begin{pmatrix}x^2\\0\end{pmatrix}\begin{pmatrix}y^2\\0\end{pmatrix}\begin{pmatrix}xy\\0\end{pmatrix}\begin{pmatrix}0\\y^2\end{pmatrix}$$

$$L_J = Jh(y) - Dh(y)Jy \quad (\text{for ex. 1})$$

To compute h_2, first identify L_J, then construct G_2 s.t. $H_2 = L_J(H_2) \oplus G_2$, calculate $h_2 \in G_2$.

<u>General procedure</u>:

1) $\dot{x} = Jx + F \implies \dot{x} = \underset{\uparrow H_2}{Jx + F_2(x)} + \underset{\uparrow H_3}{F_3(x)} + O(|x|^4)$ $\underline{o(x^2)}$

$$\begin{cases}\dot{x}_1 = x_2 + \oplus \\ \dot{x}_2 = x_1^2 + \dots\end{cases}$$

2) $x = y + h(y), \quad h_2(y) \in H_2$

$$\Rightarrow \dot{x} = Jx + \underset{\uparrow G_2}{\tilde{F}_2(x)} + \underset{\uparrow H_3}{F_3^*(x)} + O(|x|^4) \qquad \begin{cases}\dot{x}_1 = x_2 + o(x^3)\\ \dot{x}_2 = x_1^2 + \dots\end{cases}$$

3) $x = y + h(x), \quad h(x) = h_3(x) \in H_3$

$$\dot{x} = Jx + \underset{\uparrow G_2}{\tilde{F}_2(x)} + \underset{\uparrow G_3}{\tilde{F}_3(x)} + O(|x|^4) \quad \text{~~ect~~ etc.}$$

$$\begin{cases}\dot{x}_1 = x_2 + O(x^4)\\ \dot{x}_2 = x_1^2 + \dots\end{cases}$$

Since $\text{Span}\begin{pmatrix}x^2\\0\end{pmatrix}\begin{pmatrix}0\\x^2\end{pmatrix} = \text{Span}\begin{pmatrix}0\\xy\end{pmatrix}\begin{pmatrix}0\\x^2\end{pmatrix} \Rightarrow$ normal form is not unique.

Summary: $\dot{x} = Jx + F(x)$

1) $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad F(x) = \begin{pmatrix} ax^2 + bxy + cy^2 \\ dx^2 + exy + fy^2 \end{pmatrix} + O(|x|^3)$ $\quad |x| \to 0$

can be put into the normal form

$$\begin{cases} \dot{x} = \underbrace{y + O(|x|^3)}_{\to y} \\ \dot{y} = dx^2 + (e+2a)xy + O(|x|^3) \end{cases}$$

then by $y \to y + O(|x|^3)$

$$\begin{cases} \dot{x} = y \\ \dot{y} = dx^2 + (e+2a)xy + O(|x|^3) \end{cases}$$

nonhyperbolic equilibrium at $(0,0)$

if $d \neq 0$, $e+2a \neq 0 \Rightarrow O(|x|^3)$ terms do not influence the phase behavior

$\Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = x^2 \pm xy \end{cases}$ gives a "close" system with all qualitative features of the above.

$\begin{cases} \dot{x} = y \\ \dot{y} = \mu_1 + \mu_2 y + x^2 \pm xy \end{cases}$ ← "universal unfoldings"

2) if J has the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ← 2 eig. v. with $Re(\lambda_i) = 0$

$$\begin{aligned} \dot{x} &= -y + O(|x|^2) \\ \dot{y} &= x + O(|x|^2) \end{aligned} \quad \Longrightarrow$$

normal form $\begin{cases} \dot{x} = -y + (ax - by)(x^2+y^2) + O(|x|^4) \\ \dot{y} = x + (ay + bx)(x^2+y^2) + O(|x|^4) \end{cases}$

weak focus at $(0,0)$