Math 677
Lecture 17

Liapunov fct $V(x,y)$:
to prove stability, necessary to have
$V=0$ at x_0 - crit. pt.
$V>0$ for all $\vec{x} \neq \vec{x}_0$

to prove unstable crit. pt. at x_0,
can choose V s.t.
$V(x_0)=0$, $V(\vec{x})>0$ for
some $\vec{x}$ arbitrary close to x_0 (can be a subset of $N_\delta(x_0)$)

Ex. $\begin{cases} \dot{x} = x^3 + yx^2 \\ \dot{y} = -y + x^3 \end{cases}$ Prove that $(0,0)$ - unstable by Lyapunov method.

$$Df = \begin{pmatrix} 0 & x^2 \\ 0 & -1 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{non-hyperbolic pt.}$$

$$V = \frac{1}{2}(x^2 - y^2) \qquad \dot{V} = x\cdot\dot{x} - y\cdot\dot{y}$$
$$= x(x^3+yx^2) - y(-y+x^3)$$
$$= x^4 + \cancel{yx^3} + y^2 - \cancel{yx^3} = x^4 + y^2 > 0$$

Last time: CMT for the case of $Re(\lambda_i) \leq 0$

Now: general form of CMT.

Thm. (Local Central Manifold Thm)

$E \subset \mathbb{R}^n$-open, $\{0\} \in E$, $f \in C^1(E)$, $f(0)=0$

$Df(0) = \mathrm{diag}[C, P, Q]$

$Re(\lambda_i)=0$; $Re(\lambda_i)<0$; $Re(\lambda_i)>0$

$\Rightarrow \exists\, h_1(x) \in C^1, h_2(x) \in C^1$ s.t. in $N_\delta\{0\}$

$$\begin{cases} Dh_1[Cx + F(x,h_1,h_2)] - Ph_1 - G(x,h_1,h_2) = 0 \\ Dh_2[Cx + F(x,h_1,h_2)] - Qh_2 - H(x,h_1,h_2) = 0 \end{cases}$$

s.t. $\dot{x} = f(x)$ can be written in the form

$$\begin{cases} \dot{x} = Cx + F(x,y,z) \\ \dot{y} = Py + G(x,y,z) \\ \dot{z} = Qz + H(x,y,z) \end{cases}$$ top. conjugate

to the system $$\begin{cases} \dot{x} = Cx + F(x,h_1,h_2) \\ \dot{y} = Py \\ \dot{z} = Qz \end{cases}$$ in $N_\delta\{0\}$.

$\mathbb{R}^2$

Hyperbolic pts: nodes, saddles, foci, centers, center-focus

Non-hyperbolic pts: cusp, saddle-node, elliptic domain

Consider $A = Df(x_0) \Rightarrow A$ has at least one zero eigenvalue

Case 1. $\operatorname{tr} A \neq 0$ $(\lambda_1 = 0, \lambda_2 \neq 0)$

$$\dot{x} = f(x) \sim \begin{cases} \dot{x} = p_2(x,y) \\ \dot{y} = y + q_2(x,y) \end{cases}$$

$p_2, q_2 \in C^\infty$, start with quadratic terms in expansion.

Claim 1: $\{0\}$ - isolated pt of $\dot{x} = f(x)$.

$y = \varphi(x)$: solution to $y + q_2(x,y) = 0$ in $N_\delta(0)$.

$\Psi = p_2(x, \varphi(x)) = \underset{\neq 0}{a_m} x^m + \dots$, $m \geq 2$

$\Rightarrow$ (1) m-odd, $a_m > 0$ $\Rightarrow$ $\{0\}$ - unstable node

(2) m-odd, $a_m < 0$ $\Rightarrow$ $\{0\}$ - saddle

$\Rightarrow$ (3) m-even $\Rightarrow$ $\{0\}$ - saddle-node

Case 2. $\operatorname{tr} A = 0$ $(\lambda_1 = 0, \lambda_2 = 0)$

$$\dot{x} = f(x) \sim \begin{cases} \dot{x} = y \\ \dot{y} = a_k x^k [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 R(x,y) \end{cases}$$

Normal form

h, g, R - analytic

$h(0) = g(0) = 0$, $k \geq 2$

$a_k \neq 0$, $n \geq 1$

Claim 2: $k = 2m+1$, $m \geq 1$

$\lambda = b_n^2 + 4(m+1) a_k$

if $a_k > 0$ $\Rightarrow$ $\{0\}$ - saddle

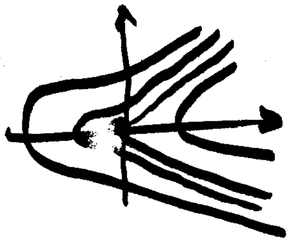
if $a_k < 0$ $\Rightarrow$ (1) $b_n = 0$ $\Rightarrow$ focus or center

(2) $b_n \neq 0$, $n > m$; $n = m$, $\lambda < 0$ $\Rightarrow$ focus or center

(3) $b_n \neq 0$, n-even, $n < m$; $b_n \neq 0$, n-even, $n = m$, $\lambda \geq 0$ $\Rightarrow$ node

$\longrightarrow$ (4) $b_n \neq 0$, n-odd $n < m$; $b_n \neq 0$ n-odd $n = m$, $\lambda \geq 0$ $\Rightarrow$ elliptic domain

Claim 3: $k = 2m,\ m \geq 1$.



if (1) $b_n = 0$, $b_n \neq 0,\ n \geq m$ $\Rightarrow$ cusp

(2) $b_n \neq 0,\ n < m$ $\Rightarrow$ saddle-node

Normal form theory.

$\dot{x} = f(x) \rightsquigarrow \dot{x} = Jx + F(x)$

x_0 – nonhyperbolic pt

J – Jordan form of A

By ~~CPM~~ CMT only C-part of J matters ($\dim C < \dim A$)

CMT – Centrale Manifold Theorem

reduced system to a lower-dim case

Normal form theory: reduces $\dot{x} = Jx + F(x)$ to a system with a simple form of $F(x)$.

This is done by a nonlinear transformation $x = y + h(y)$, $h(y) = O(|y|^2)$, $|y| \to 0$.

This transformation is almost like identity close to $\{0\}$ $\Rightarrow$ we end up with a top. conjugate system.