Math 677.
Lecture 16.

$\dot{x} = f(x, t)$ (2) $\qquad \dot{x} = Ax$ (1)

$A = Df(0)$

$\{0\}$ - equilibrium

To study stability, it is useful to use substitution $x = r\cos\theta$ in 2d
$y = r\sin\theta$

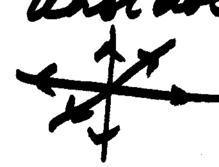
2 useful relations: $r\cdot\dot{r} = x\cdot\dot{x} + y\cdot\dot{y}$

$r^2\dot{\theta} = x\dot{y} - y\cdot\dot{x}$

<u>Cases:</u>

1) $\exists \delta > 0$ s.t. $0 < r_0 < \delta$, $\theta_0 \in \mathbb{R}$
$r(t, r_0, \theta_0) \to 0$ as $t \to \infty$
$\exists \lim_{t\to\infty} \theta(t, r_0, \theta_0)$ $\Rightarrow$ stable node

$\exists \delta > 0$ s.t. $0 < r_0 < \delta$, $\theta_0 \in \mathbb{R}$
$r(t, r_0, \theta_0) \to 0$ as $t \to -\infty$ $\Rightarrow$ unstable node
$\exists \lim_{t\to-\infty} \theta(t, r_0, \theta_0)$



2) $\exists \delta > 0$ s.t. $0 < r_0 < \delta$, $\theta_0 \in \mathbb{R}$
$r(t, r_0, \theta_0) \to 0$ as $t \to \infty$ $\Rightarrow$ stable focus
$|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$

$r(t, r_0, \theta_0) \to 0$ as $t \to -\infty$
$|\theta(t, r_0, \theta_0)| \to \infty$ $\to$ unstable focus

3) $\exists \Gamma_1, \Gamma_2 \to 0$ as $t \to \infty$
$\Gamma_3, \Gamma_4 \to 0$ as $t \to -\infty$

4) $\{0\}$ is such that $\forall \delta > 0$ $N_\delta(0)$ contains a closed curve (trajectory) $\Rightarrow$ center.

5) center-focus
sequence of $\Gamma_{n+1} \subset \mathrm{Int}(\Gamma_n)$, $\Gamma_n \xrightarrow[n\to\infty]{} 0$
trajectories between Γ_n & Γ_{n+1}
spiral toward Γ_n or Γ_{n+1}.

<u>Thm.</u> (Bendixson)
$E \subset \mathbb{R}^2$ - open, $\{0\} \in E$
$f \in C^1(E)$
$\{0\}$ - isolated crit. pt. of (2)
$\Rightarrow$ Either every $N(0)$ contains a closed solution curve containing $\{0\}$, or there is a trajectory approaching $\{0\}$ as $t \to \pm\infty$.

<u>Thm.</u> $P(x,y)$, $Q(x,y)$ - ~~any~~ analytic in x, y in $E \subset \mathbb{R}^2$, $\{0\} \in E$

Suppose Taylor expansions of P, Q about $\{0\}$ begin with m-th degree terms P_m, Q_m, $m \geq 1$

Then system $\dot{x} = P(x,y)$
$\dot{y} = Q(x,y)$
has trajectories that approach $\{0\}$ as $t \to \infty$ either in a spiral or by a definite direction $\theta = \theta_0$
If $xQ_m - yP_m \neq 0$ then all directions satisfy
$\cos\theta_0 Q_m - \sin\theta_0 P_m = 0$

Moreover, if one trajectory spirals toward $\{0\}$ as $t \to \infty$, then in $N\{0\} \setminus \{0\}$ all of them will spiral toward $\{0\}$.

Summary of nonlinear stability:

	Linear (1)		Nonlinear (2)
1) $f \in C^1(E)$	node	$\longleftrightarrow$	node / focus
	focus	$\longleftrightarrow$	focus
	saddle	$\longleftrightarrow$	saddle
	center	$\longleftrightarrow$	center / focus / center-focus

2) $f \in C^1(E)$,
f-symmetric in x-axis, y-axis
((2) does not change if substitution $(t,x) \to (t,-x)$ or $(t,y) \to (t,-y)$ is ~~are~~ made)
same, except for:

center	$\longleftrightarrow$	center / ~~focus~~

3) $f \in C^2(E)$	node	$\longleftrightarrow$	node
	focus	$\longleftrightarrow$	focus
	saddle	$\longleftrightarrow$	saddle
	center	$\longleftrightarrow$	center / focus / center-focus
4) $f \in C^\infty(E)$	center	$\longleftrightarrow$	center / focus

$$\dot{x} = y, \quad \dot{y} = -x \qquad (1)$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}, \quad A = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$(0,0)$ - center for (1)

Ex. 1

$$\begin{cases}\dot{x} = y + \sqrt{x^2+y^2}\cdot x\\ \dot{y} = -x + \sqrt{x^2+y^2}\cdot y\end{cases}$$

$r^2 = x^2+y^2$

$\theta = \arctan\left(\frac{y}{x}\right)$

$$r\cdot\dot{r} = x\cdot\dot{x} + y\cdot\dot{y}$$
$$= \cancel{xy} + \sqrt{x^2+y^2}\,x^2 - \cancel{xy} + \sqrt{x^2+y^2}\cdot y^2$$
$$= \sqrt{x^2+y^2}(x^2+y^2) = r^3$$

$$\Rightarrow \underbrace{\dot{r} = r^2} \qquad r = \frac{1}{r_0 - t}, \quad r_0 = \frac{1}{r(0)}$$

$r(t) \to \infty$ as $t \to r_0$.

$r(t) \to 0$ as $t \to -\infty$.

$$r^2\dot{\theta} = x\dot{y} - y\cdot\dot{x} = -x^2 + \cancel{xy\sqrt{x^2+y^2}} - y^2 - \cancel{xy\sqrt{x^2+y^2}}$$
$$= -r^2 \Rightarrow \underbrace{\dot{\theta} = -1}$$

$\theta = -t + C \to \infty$ as $t \to -\infty$

$\Rightarrow \{0\}$ - **unstable focus.**

Ex. 2 $\begin{cases}\dot{x} = y + \sqrt{x^2+y^2}\cdot x\\ \dot{y} = -x - \sqrt{x^2+y^2}\cdot y\end{cases}$ $\Rightarrow \dot{r} = 0 \Rightarrow$ **center**

Ex. 3 $\begin{cases}\dot{x} = y - (x^2+y^2)\cdot x\\ \dot{y} = -x - y(x^2+y^2)\end{cases}$ $\quad \frac{1}{2}\dot{r^2} = -(r^2)^2$

asymp. stable focus $\quad r^2(t) = \frac{c}{1+2ct} \xrightarrow[t\to\infty]{} 0$

Ex. 4 $\begin{cases}\dot{x} = y + xy^3 - y^7\\ \dot{y} = x + xy^2 - y^6\end{cases}$ $\quad (t, y) \to (-t, -y)$

z

$$\left.\begin{aligned}-\dot{x} &= -y + xy^3 - y^7\\ \dot{y} &= -x + xy^2 - y^6\end{aligned}\right\} \quad \cancel{\begin{aligned}\dot{z} &= y + zy^3 - y^7\\ \dot{y} &= z + zy^2 - y^6\end{aligned}}$$

same as original

~~Same for $(t,y) \to (t,-y)$~~ $\Longrightarrow$

we have $f(x,t)$ - symmetric in y-axis ~~and y axis~~

$\Rightarrow \{0\}$ - center.

Center Manifold Theory.

Hartman-Grobman Thm: $\{x_0\}$ - hyperbolic $\Rightarrow$ qualitatively (1) is top. equivalent to (2).

Thm. $f \in C^r(E)$, $E \subset \mathbb{R}^n$ - open, $\{0\} \in E$, $r \geq 1$

$f(0) = 0$, $Df(0) = 0$.

has c eig. values $Re(\lambda_i) = 0$ $s + c = n$

s eig. values $Re(\lambda_i) < 0$

Then $\dot{x} = f(x)$ can be written as

$$\dot{x} = Cx + F(x,y), \quad C - \text{eig.v. } Re(\lambda_i) = 0$$
$$\dot{y} = Py + G(x,y), \quad P - \text{eig.v } Re(\lambda_i) < 0$$

$F(0) = G(0) = 0$, $DF(0) = D\cancel{G}(0) = 0$.

$\exists \delta > 0$, $h \in C^r(N_\delta(0))$ that defines local center manifold:

$$W^c_{loc}(0) = \{x \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x), |x| < \delta\}$$

and h satisfies

$$(CM) \qquad Df(x)[Cx + F(x,h)] - Ph - G(x,h) = 0 \qquad |x| < \delta$$

The flow on CM $W^c_{loc}(0)$ is then

$$\dot{x} = Cx + F(x, h(x)), \quad x \in \mathbb{R}^c, \quad |x| < \delta.$$

Ex. $\begin{cases} \dot{x} = x^2 y - x^5 \\ \dot{y} = -y + x^2 \end{cases}$ $\quad A = Df(0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & P \end{pmatrix}$

$C = [0]$, $P = [-1]$

$c = s = 1$

$\begin{cases} \dot{x} = Cx + F \\ \dot{y} = Py + G \end{cases}$ $\quad F = \underline{x^2 y - x^5}$, $\quad G = x^2$

To find h: $h(x) = ax^2 + bx^3 + O(x^4)$

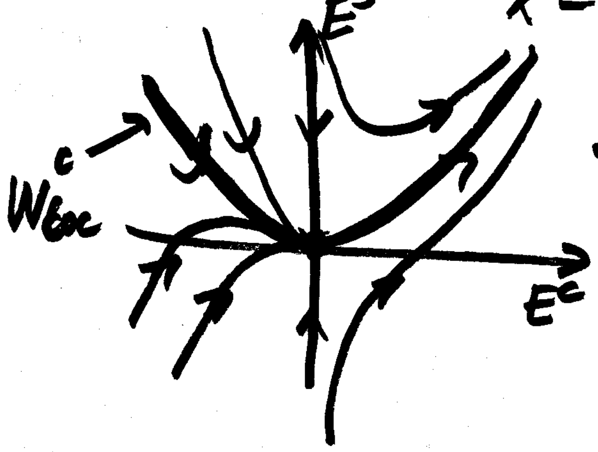
$Dh(x) = 2ax + 3bx^2 + O(x^3)$

Plug into (CM): $(2ax + 3bx^2 + \dots)(ax^4 + bx^5 + \dots - x^5) + ax^2 + bx^3 + \dots - x^2 = 0$

$a = 1$, $b = 0$, $c = 0$ $\Rightarrow$ $h = x^2 + O(x^5)$, $y = h(x)$

Flow equation:

$\dot{x} = Cx + F(x, h) = x^4 + O(x^5)$



saddle-node