Lecture 15.
Math 677.

Stability of nonlinear systems:

Ex. $\begin{cases} \dot{x} = -x^3 + 2y^3 \\ \dot{y} = -2xy^2 \end{cases}$ $\quad Df(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Linearization doesn't give anything in terms of $(0,0)$ - critical point stability

$V = \frac{1}{2}(x^2+y^2)$ $\left[\underset{x \neq x_0}{V(x) \geq 0}, \underset{\text{for } x = x_0}{V(x) = 0 \text{ only}} \right]$

$\dot{V} = x \cdot \dot{x} + y \cdot \dot{y} =$

$= x(-x^3+2y^3) + y(-2xy^2) = -x^4 < 0$

$\Rightarrow (0,0)$ - asymp. stable

Liapunov function helped to classify this equilibrium.

orbits decrease the values of V.

Thm. E - open $\subset \mathbb{R}^n$ $\quad x_0 \in E$

$f \in C^1(E) \quad f(x_0) = 0$

$\exists$ real-valued function $V \in C^1(E)$

s.t. $V(x_0) = 0$, $V(x) > 0$ if $x \neq x_0$

Then (a) if $\dot{V}(x) \leq 0 \ \forall x \in E \Rightarrow x_0$ - stable

(b) if $\dot{V}(x) < 0 \ \forall x \in E \Rightarrow x_0$ - asymp. stable

(c) if $\dot{V}(x) > 0 \ \forall x \in E \Rightarrow x_0$ - unstable

Proof. WLOG $x_0 = 0$.

(a) $\dot{V}(x) \leq 0 \ \forall x \in E$.

To show: $\forall \varepsilon > 0 \ \exists \delta$ s.t. $\forall x \in N_\delta(0)$ $\varphi_t(x) \in N_\varepsilon(0)$ i.e. $x_0 = 0$ is stable

Choose $\varepsilon > 0$ s.t. $\overline{N_\varepsilon(0)} \subset E$.

Fix $m_\varepsilon = \min\limits_{S_\varepsilon} V(x)$, where $S_\varepsilon = \{x \mid |x| = \varepsilon\}$

Since $V(x) > 0 \ \forall x \in E \Rightarrow m_\varepsilon > 0$

V-continuous, $V(0) = 0 \Rightarrow \exists \delta > 0$ s.t. $|x| < \delta \Rightarrow V(x) < m_\varepsilon$

Since $\dot{V}(x) \leq 0$ on E ($V(x) \searrow$ along trajectories of $\dot{x} = f(x)$ (2)

$\Rightarrow$ φ_t-flow of (2)

$\forall x_0 \in N_\delta(0), t \geq 0 \Rightarrow V(\varphi_t(x_0)) \leq V(x_0) < m_\varepsilon$

Suppose $\exists t_1 > 0$ s.t. $|\varphi_{t_1}(x_0)| = \varepsilon$ i.e. $\varphi_{t_1}(x_0) \in S_\varepsilon$

Then since $m_\varepsilon = \min\limits_{S_\varepsilon} V \Rightarrow V(\varphi_{t_1}(x_0)) \geq m_\varepsilon$

So for $|x_0| < \delta$, $t \geq 0 \Rightarrow |(\varphi_t(x_0))| < \varepsilon$

$\Rightarrow x_0$-stable point.

(b) if $\dot{V}(x) < 0$ then x_0-stable from (a).

To show: $t_k \to \infty$ $\varphi_{t_k}(x_0) \to 0$ $\forall x_0 \in E$. (asymp. stability)

$\exists$ subsequence $\{\varphi_{t_k}(x_0)\} \to x^*$, then $x^* = 0$. (proof by contradiction).

(c) $\dot V(x) > 0$ $V \uparrow$ along trajectories

φ_t - flow

then $\forall \delta > 0$, $x_0 \in N_\delta(0) \setminus \{0\}$

$V(\varphi_t(x_0)) > V(x_0) > 0$

$V(\varphi_t(x_0)) > V(x_0) > 0 \quad \forall t \geq 0$

$\inf \dot V(\varphi_t(x_0)) = m > 0$ since $\dot V$ - pos. def.

$\Rightarrow V(\varphi_t(x_0)) - V(x_0) \geq mt \quad \forall t \geq 0$

$V(\varphi_t(x_0)) \geq mt > M$ for suff. large t

$\Rightarrow$ x_0 - <u>unstable</u> since $\varphi_t(x_0)$ lies outside $N_\delta(0)$ for any δ.

2.10. Saddles, nodes etc. for nonlinear case

$\dot x = f(x,t)$ (2), $\quad \dot x = Ax$ (1), $A = Df(0)$, $\quad \begin{cases} \dot x = P(x,y) \\ \dot y = Q(x,y) \end{cases}$ (3)

So far:

sink if $Re(\lambda_i) < 0$

Source if $Re(\lambda_i) > 0$ $\quad \Leftarrow$ $\begin{cases} \text{saddles} \\ \text{nodes} \\ \text{foci} \\ \text{centers} \end{cases}$ <u>equilibria</u>

In 2D, polar coordinates are useful for classifying equilibria:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{array}{l} r^2 = x^2 + y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \end{array} \Rightarrow \begin{cases} \dot r = P\cos\theta + Q\sin\theta \\ r\dot\theta = Q\cos\theta - P\sin\theta \end{cases}$$

$\dot r = 0 \Rightarrow$ center

$\dot r < 0 \Rightarrow r(t) \to 0$ as $t \to \infty$ $\leftarrow$ stable

$\dot r > 0 \Rightarrow r(t) \to 0$ as $t \to -\infty$ $\leftarrow$ unstable

Def 1) $\{0\}$ - <u>center</u> of (2) if $\exists \delta > 0$ s.t. every solution of (2) in $N_\delta(0)\setminus\{0\}$ is a closed curve containing $\{0\}$.

2) $\{0\}$ - <u>center-focus</u> of (2) if $\exists$ sequence of closed solution curves Γ_n with $\Gamma_{n+1} \subset \mathrm{int}(\Gamma_n)$ s.t. $\Gamma_n \to 0$ as $n \to \infty$ and every trajectory between Γ_n and Γ_{n+1} spirals toward Γ_n or Γ_{n+1} as $t \to \infty$.

3) $\{0\}$ - stable (unstable) <u>node</u> if $\exists \delta > 0$ s.t. $0 < r_0 < \delta$, $\theta_0 \in \mathbb{R}$ $r(t, r_0, \theta_0) \to 0$ as $t \to \infty$ and $(t \to -\infty$ resp.$)$ $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists $\left(\lim_{t\to-\infty} \theta(t, r_0, \theta_0)\right)$

<u>proper node</u> - every ray through $\{0\}$ is tangent to some trajectory of (2).

4) $\{0\}$ - (top.) <u>saddle</u> if $\exists \Gamma_1, \Gamma_2 \to 0$ as $t \to \infty$ and $\exists \Gamma_3, \Gamma_4 \to 0$ as $t \to -\infty$ and $\exists \delta > 0$ s.t. all other trajectories starting in $N_\delta(0)\setminus\{0\}$ leave this neighborhood as $t \to \pm\infty$.