Lecture 14.

Math 677.

$\dot{x} = f(x)$ (1) $\quad \varphi_t$ - flow

Case 1. x_0 - hyperbolic equilibrium in $\mathbb{R}^n$

Th 1) x_0 - asymp. stable iff $Re(\lambda_i) < 0$ $i = 1, \dots n$
(sink)

2) x_0 - unstable if $\exists k$ s.t. $Re(\lambda_k) > 0$
(source or saddle) $\quad 1 \le k \le n$.

Case 2. x_0 - nonhyperbolic equilibrium

1) x_0 - stable if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.
$\forall x \in N_\delta(x_0)$, $t \geq 0$ $\quad \varphi_t(x) \in N_\varepsilon(x_0)$.

2) x_0 - unstable - otherwise

3) x_0 - asymp. stable if it is stable
and $\exists \delta > 0$ s.t. $\forall x \in N_\delta(x_0)$
$$\lim_{t \to \infty} \varphi_t(x) = x_0$$

Linear systems in $\mathbb{R}^2$:

Stable node, Stable focus $\rightarrow$ asymp. stable

Unstable node, unstable focus, saddle $\rightarrow$ unstable

Center or $\lambda = 0$ $\rightarrow$ stable, not asymp. stable

Nonlinear systems:

By Hartman-Grobman theorem,

if x_0 - hyperbolic, x_0 - sink $\rightarrow$ asymp. stable

x_0 - hyperbolic, x_0 - source, saddle $\Rightarrow$ unstable

One case remaining: when $Re(\lambda_k)=0$ for some k.

Claim. if x_0 - stable equilibrium of (1) (i.e. $Df(x_0)$ has at least one $Re(\lambda_k)=0$) then there is no eigenvalue λ of $Df(x_0)$ with $Re(\lambda)>0$.

Non-hyperbolic case is best analysed by Lyapunov functions.

Def. $V: \mathbb{R}^n \to \mathbb{R}$

$f \in C'(E)$, $V \in C'(E)$, φ_t - flow of $\dot{x}=f(x)$

$\forall x \in E$ the derivative of $V(x)$ along the solution $\varphi_t(x)$ is given by

$$\dot{V}(x) = \frac{d}{dt} V(\varphi_t(x))\Big|_{t=0} = DV(x)\cdot f(x)$$

Observations:

1) $\dot{V}(x)<0$ in E $\Rightarrow$ $V(x)$ decreases along the trajectory of (1)

2) $\dot{V}(x) \le 0$ in $\mathbb{R}^2$; $\dot{V}(x)=0 \Leftrightarrow x=0$.

then $V(x)=c$ for some $c>0$ is a family of closed curves enclosing $(0,0)$ s.t. the curves cross the trajectories from exterior to interior as time $\nearrow$.

So $(0,0)$ is asymp. stable.

__Theorem.__ $E \subset \mathbb{R}^n$ - open, $\{x_0\} \in E$

$f \in C'(E)$ $f(x_0) = 0$.

Suppose $\exists$ real-valued function $V \in C'(E)$

s.t. $V(x_0) = 0$

$V(x) > 0$ if $x \neq x_0$.

Then (a) if $\dot{V}(x) \leq 0$ $\forall x \in E \Rightarrow x_0$ - stable

(b) if $\dot{V}(x) < 0$ $\forall x \in E \setminus \{x_0\} \Rightarrow x_0$ - asymp. stable

(c) if $\dot{V}(x) > 0$ $\forall x \in E \setminus \{x_0\} \Rightarrow x_0$ - unstable

(d) if $\dot{V}(x) = 0$ $\forall x \in E \Rightarrow$ trajectories of (1) lie on surfaces defined by $V(x) = c$.

($V(x)$ is called Lyapunov function).

__Ex. 1.__ $\begin{cases} \dot{x} = -x + 2y^2 \\ \dot{y} = -2y + x^3 y \end{cases}$ Show: $(0,0)$ - asymp. stable

$$Df(0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow (0,0) \text{ - sink} \Rightarrow \text{asymp. stable}$$

$V(x,y) = x^2 + y^2 \leftarrow$ consider this function

$$\dot{V}(x,y) = 2x \cdot \dot{x} + 2y \cdot \dot{y} = 2x(-x + 2y^2) + 2y(-2y + x^3 y) =$$

$$= -2x^2 + 4xy^2 - 4y^2 + 2y^2 x^3 =$$

$$= -2x^2 - 2y^2(2 - 2x - x^3) \quad \text{if } x < \tfrac{1}{2}$$

$$< -2(x^2 + y^2) < 0 \Rightarrow \text{asymp. stable}$$

Ex. 2 $\begin{cases} \dot{x}_1 = -x_2^3 \\ \dot{x}_2 = x_1^3 \end{cases}$ $(0,0)$ – non-hyperbolic point

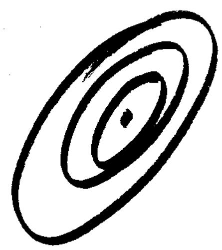
$$Df(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0.$$

$$V(x) = x_1^4 + x_2^4$$

$$\dot{V}(x) = 4x_1^3 \cdot \dot{x}_1 + 4x_2^3 \cdot \dot{x}_2 =$$

$$= 4x_1^3(-x_2^3) + 4x_2^3(x_1^3) = 0$$

* $V(x) = c^2 > 0$ defines solution curves



$\leftarrow$ $x_1^4 + x_2^4 = c^2$ $\leftarrow$ stable equilibrium at $(0,0)$

Ex. 3 $\begin{cases} \dot{x} = 2xy + x^3 \\ \dot{y} = -x^2 + y^5 \end{cases}$ $Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$V(x,y) = +x^2 + 2y^2$$

$$\dot{V}(x,y) = 2x \cdot \dot{x} + 4y \cdot \dot{y} =$$

$$= 2x(2xy + x^3) + 4y(-x^2 + y^5)$$

$$= \cancel{4x^2y} + 2x^4 - \cancel{4x^2y} + 4y^6 =$$

$$= 2x^4 + 4y^6 > 0 \Rightarrow (0,0) \text{ – unstable}$$

Ex. 4 $\begin{cases} \dot{x}_1 = -2x_2 + x_2 x_3 \\ \dot{x}_2 = x_1 - x_1 x_3 \\ \dot{x}_3 = x_1 x_2 \end{cases}$ $Df(0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$(0,0)$ – nonhyperbolic pt.

$$V = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2$$

$\dot V = DV(x) f(x) \Rightarrow$

$$\tfrac{1}{2}\dot V = \underbrace{(c_1 - c_2 + c_3)}_{=0} x_1 x_2 x_3 + \underbrace{(-2c_1 + c_2)}_{=0} x_1 x_2$$

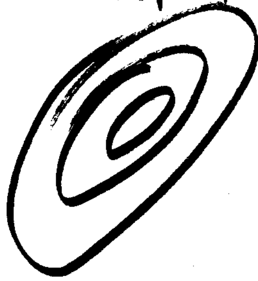
$c_2 = 2c_1$

$c_3 = c_1 > 0 \quad \Rightarrow \dot V = 0$ $\quad$ $V(x) > 0$ for $x \neq 0$.

$\Downarrow$ $\quad$ $V(x) = 0$ for $x = 0$ only.

Choose $c_1 = c_3 = 1$, $c_2 = 2$ $\quad$ $(0, 0)$ - stable, but not asymp. stable

$V = x_1^2 + 2x_2^2 + x_3^2 = c \Leftarrow$ defines solution curves around $(0, 0)$.



ellipsoid trajectories