Math 677.
Lecture 13.

Def Global stable (unstable) manifold is defined as $W^s(0) = \bigcup_{t\le 0} \varphi_t(S)$ backward in time

$W^u(0) = \bigcup_{t\ge 0} \varphi_t(U)$ respectively. forward in time

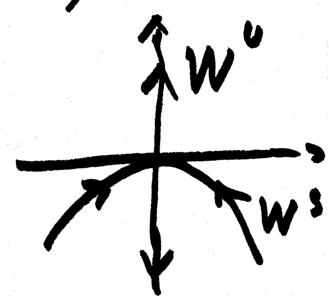
W^s, W^u - unique, invariant under φ_t.

$\forall x \in W^s(0), \ \lim_{t\to\infty} \varphi_t(x) = 0$

$\forall x \in W^u(0), \ \lim_{t\to -\infty} \varphi_t(x) = 0.$

Ex. $\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 + x_1^2 \end{cases}$ $\quad \dot{x} = Ax \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ saddle.

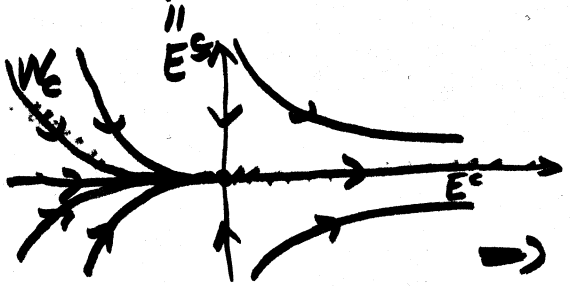
$$\begin{cases} x_1 = x_1^0 e^{-t} \\ x_2 = e^t\left(x_2^0 + \frac{1}{3}(x_1^0)^2\right) - \frac{1}{3}e^{-2t}(x_1^0)^2 \end{cases}$$



$W^u(0) = \{x_1^0 = 0\} = \{x_2\text{-axis}\}$

$W^s(0) = \{x_2^0 + \frac{1}{3}(x_1^0)^2 = 0\} = \{x_2 = -\frac{1}{3}x_1^2\}$

Ex. $\begin{cases} \dot{x}_1 = x_1^2 \\ \dot{x}_2 = -x_2 \end{cases}$ $\quad \dot{x} = Ax \quad A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$



$E^s = \{x_2\text{-axis}\}$

$E^c = \{x_1\text{-axis}\} \to W^c$ is composed of many curves tangent to E^c coinciding with x_1-axis for $x_1 > 0$.

$\Longrightarrow W^c$ is not unique.

analytical W^c is unique and coincides with E^c.

Corollary of SMT:

Under the conditions of SMT

Let S, U be stable and unstable manifolds of $\dot{x} = f(x)$ at 0, λ_j - eig. values of $Df(0)$.

$$Re(\lambda_j) < -\alpha < 0 < \beta < Re(\lambda_m)$$
$$j = 1, \dots k \qquad m = k+1, \dots n$$

$\Rightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $x_0 \in N_\delta(0) \cap S$ then $|\varphi_t(x_0)| \leq \varepsilon e^{-\alpha t}$ $\forall t \geq 0$

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $x_0 \in N_\delta(0) \cap U$ then $|\varphi_t(x_0)| \leq \varepsilon e^{\beta t}$ $\forall t \leq 0$

Thm. (CMT) Center Manifold Thm.

$\boxed{f \in C^r(E)}$, $E \subset \mathbb{R}^n$ - open, $\{0\} \in E$
$r \geq 1$

$f(0) = 0$, $Df(0)$ has λ_i - eig. values s.t.

$$Re(\lambda_i) < 0 \quad i = 1, \dots k$$
$$Re(\lambda_i) > 0 \quad i = k+1, \dots k+j$$
$$m = n-k-j \qquad Re(\lambda_i) = 0 \quad i = k+j+1, \dots n$$

$\Rightarrow$ There exists an m-dim manifold $W^c(0)$ of class C^r tangent to E^c at 0

$\leftarrow$ center manifold of $\dot{x} = Df(0)x$

$\exists W^s(0), W^u(0)$ - tangent to E^s, E^u resp.

Moreover, W^u, W^s, W^c - φ_t-invariant.

Comment: W^u, W^s, W^c can fail to be embedded manifolds in $\mathbb{R}^n$.

Ex. $\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases}$ $y = \alpha e^{1/x}$ $\leftarrow$ family of center manifolds.

Thm. (Hartman-Grobman Thm.)

(1) $\dot{x} = f(x)$ (2) $\dot{x} = Ax$, $A = Df(0)$

equilibrium x_0 translated to the origin

Def. (1), (2) - top. equivalent at x_0 (or at 0) if $\exists\ \varphi$ - homeomorphism: $U \to V$

open $\{0\} \in U$ open $\{0\} \in V$

which maps trajectories of (1) ~~into~~ in U into trajectories of (2) in V, and preserves their time-orientation.

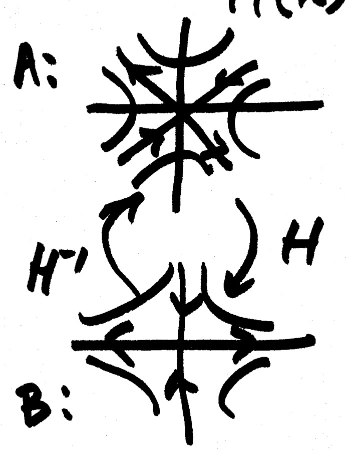
If φ preserves parameterization by time, then (1), (2) are called top. conjugate in a nbhd of $\{0\}$.

Ex. $A = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ $\dot{x} = Ax$ $\dot{y} = By$

top. conjugate

$H(x) = Rx$ s.t. $R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ rotation by 45°

$R^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$



$\Rightarrow$

$B = RAR^{-1}$ $y = Rx$ $x = R^{-1}y$

$\dot{y} = RA\underbrace{R^{-1}y}_{} = By$

$x(t) = e^{At}x_0 \Rightarrow y(t) = Re^{At}x_0 =$
$= e^{At}Rx_0 = e^{At}y_0$

Hartman-Grobman Thm

$E \subset \mathbb{R}^n$-open $\{0\} \in E$

$f \in C^1(E)$ φ_t - flow of $\dot{x} = f(x)$, $f(0) = 0$

$A = Df(0)$.

If A has no eigenvalue λ_i with $Re(\lambda_i) = 0$

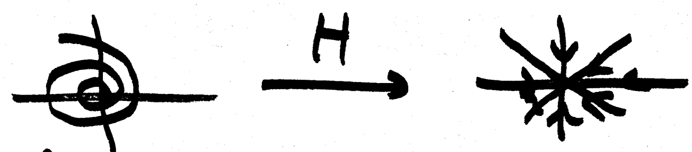
then: $\exists$ H-homeo: $\underset{\{0\}}{U} \longrightarrow \underset{\{0\}}{V}$ s.t

$\forall x_0 \in U$ $\exists \underset{\{0\}}{I_0} \subset \mathbb{R}$ s.t. $\forall t \in I_0$ $H \circ \varphi_t(x_0) = e^{At} H(x_0)$

(H maps trajectories of $\dot{x} = f(x)$ near $\{0\}$ onto trajectories of $\dot{x} = Ax$ near $\{0\}$ and preserves parameterization in time).

<u>Ex</u>. $$\begin{cases} \dot{x}_1 = -x_1 - \dfrac{x_2}{\ln\sqrt{x_1^2 + x_2^2}} \\ \dot{x}_2 = -x_2 + \dfrac{x_1}{\ln\sqrt{x_1^2 + x_2^2}} \end{cases} \rightsquigarrow \begin{cases} \dot{y}_1 = -y_1 \\ \dot{y}_2 = -y_2 \end{cases}$$

$(0,0)$ - spiral — stable node



H - homeo but not a diffeomorphism

Reason: $f \in C^1(E)$ but not $C^2(E)$

$f \in C^2(E)$.

<u>Thm</u> (Hartman)

$E \subset \mathbb{R}^n$ - open $\{0\} \in E$

$f \in C^2(E)$, $Re\,\lambda_i \neq 0$ for $Df(0)$ eigenvalues

$\Rightarrow \exists$ C^1-diffeomorphism H mapping trajectories of (1) onto (2) and back:

$\Rightarrow \quad H: \underset{\text{open}}{U_{x_0}} \longrightarrow \underset{\text{open}}{V}$

s.t. $\forall x \in U \;\exists I(x) \subset \mathbb{R}$ s.t.

$\forall x \in U \;\forall t \in I \quad H \circ \varphi_t(x) = e^{At} H(x).$