Math 677
Lecture 12

Thm. (Stable Manifold Thm). – local result

$E \subset \mathbb{R}^n$ – open $\{0\} \in E$

$\boxed{f \in C^1(E)}$, φ_t – flow of $\dot{x} = f(x)$

Let $f(0) = 0$ and $A = Df(0)$ has $\lambda_1, \dots \lambda_n$ – eigenvalues with $Re(\lambda_i) < 0$, $i = 1, \dots k$
$Re(\lambda_i) > 0$ $i = k+1, \dots n$

$\Rightarrow$ $\exists$ k-dim diff. manifold S, tangent to E^s (for $\dot{x} = Ax$) at $\{0\}$ s.t.

$\forall t \geq 0$ $\varphi_t(S) \subset S$ and $\forall x_0 \in S$ $\lim_{t \to \infty} \varphi_t(x_0) = 0$

Similarly, $\exists$ $(n-k)$-dim diff. manifold U tangent to E^u at $\{0\}$ s.t.

$\forall t \leq 0$ $\varphi_t(U) \subset U$ and $\forall x_0 \in U$ $\lim_{t \to -\infty} \varphi_t(x_0) = 0$.

Proof

$$B = C^{-1}AC = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \qquad \underline{\dot{y} = By + G(y)}$$

$$\underline{y = C^{-1}x}$$

$$U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} \qquad V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$$

$\dot{U} = BU$, $\dot{V} = BV$, $e^{Bt} = U(t) + V(t)$

Choose $\alpha > 0$ s.t. $+Re(\lambda_i) < -\alpha < 0$ $i = 1, \dots k$

$\Rightarrow \exists K > 0$ $\sigma > 0$ s.t. $\|U(t)\| \leq K e^{-(\alpha+\sigma)t}$, $t \geq 0$

$\|V(t)\| \leq K e^{\sigma t}$, $t \leq 0$.

Consider integral equation:

$$(*) \quad \left[u(t,a) = U(t)a + \int_0^t U(t-s)\, G(u(s,a))\, ds - \int_t^\infty V(t-s)\, G(u(s,a))\, ds \right]$$

If $u(t,a)$ - solution of ⊛ $\Rightarrow$ $u(t,a)$ solves $\dot{y} = By + G(y)$ for all t (check by direct calculation)

By successive approximations,

$$u^{(0)}(t,a) = 0$$

$$u^{(j+1)}(t,a) = U(t)a + \int_0^t U(t-s)\,G(u^{(j)}(s,a))\,ds - \int_t^{\infty} V(t-s)\,G(u^{(j)}(s,a))\,ds$$

Wanted: $\{u^{(j)}(t,a)\} \to u^{t}(t,a)$

By induction, we show that $\left|u^{(j)}(t,a) - u^{(j-1)}(t,a)\right| \le \dfrac{K|a|e^{-\alpha t}}{2^{j-1}}$, $t \ge 0$, $j = 1, 2, \ldots$

Basis of induction: $j=1$, $t \ge 0$ $\Rightarrow$ $|u^{(1)}(t,a)| \le K|a|e^{-\alpha t}$ trivial

Suppose it holds for $j=m$

For $j=m+1$: use Lipschitz condition for G

$$\left|u^{(m+1)}(t,a) - u^{(m)}(t,a)\right| \le$$

$$\le \varepsilon \int_0^t \|U(t-s)\| \cdot \left|u^{(m)}(s,a) - u^{(m-1)}(s,a)\right| ds + \varepsilon \int_t^{\infty} \|V(t-s)\| \cdot \left|u^{(m)}(s,a) - u^{(m-1)}(s,a)\right| ds$$

$$\le \varepsilon \int_0^t K e^{-(\alpha+\sigma)(t-s)} \frac{K|a|e^{-\alpha s}}{2^{m-1}}\,ds + \varepsilon \int_t^{\infty} K e^{\sigma(t-s)} \frac{K|a|e^{-\alpha s}}{2^{m-1}}\,ds$$

$$\le \frac{\varepsilon K^2 |a| e^{-\alpha t}}{\sigma 2^{m-1}} + \frac{\varepsilon K^2 |a| e^{-\alpha t}}{\sigma \cdot 2^{m-1}}$$

take $\left(\varepsilon \frac{K}{\sigma} < \frac{1}{4}\right)$ $\boxed{\varepsilon < \frac{\sigma}{4K}}$

$$< \frac{K|a|e^{-\alpha t}}{2^m}$$ $\Rightarrow$ induction step is done.

Lipshitz condition for G holds if
$$K|a| < \frac{\delta}{2} \Rightarrow \boxed{|a| < \frac{\delta}{2K}}$$

Now: $\{u^{(j)}(t,a)\}$ – Cauchy sequence, since
$$|u^{(n)}(t,a) - u^{(m)}(t,a)| \le \sum_{j=N}^{\infty} |u^{(j+1)}(t,a) - u^{(j)}(t,a)| \le \quad n,m \ge N$$
$$\le K|a| \sum_{j=N}^{\infty} \frac{1}{2^j} = \frac{K|a|}{2^{N-1}} \xrightarrow[N\to\infty]{} 0$$

Cauchy sequence of cts fcts $u^{(j)}(t,a) \Rightarrow$
Uniform convergence $\{u^{j}(t,a)\} \xrightarrow[j\to\infty]{} u(t,a)$ $\forall t \ge 0$.

Since $u(t,a)$ satisfies (*) for all $|a| < \frac{\delta}{2K}$
it solves $\dot{y} = By + G(y)$.

$G \in C'(\tilde{E})$, where $\tilde{E} = C^{-1}(E)$ (since $f \in C'(E)$)

So $u^{(j)}(t,a)$ - differentiable for $t \ge 0$.
$|a| \le \frac{\delta}{2K}$

By uniform convergence,
(**) $|u(t,a)| \le 2K|a|e^{-\alpha t}$, $t \ge 0$, $|a| \le \frac{\delta}{2K}$

Can be shown: if we pick

initial conditions
$$\begin{cases} u_j(0,a) = a_j, & j = 1, \dots k \\ u_j(0,a) = -\left(\int_0^\infty V(-s)G(u(s,a_1,\dots a_k,0))\,ds\right)_j & j = k+1, \dots n \end{cases}$$

then $\psi_j(a_1,\dots a_k) = u_j(0, a_1, \dots a_k, \underbrace{0,\dots 0}_{k+1,\dots n \text{ do not enter the solution.}})$

$j = k+1, \dots n$

$y_j = \Psi_j(y_1, \dots y_k) \quad j = k+1, \dots n$

$y_j = u_j(0, a_1 \dots a_k, 0 \dots 0)$ defines a stable diff. manifold $\tilde{S}_k$ in y-space.

for $\left(\sum_{i=1}^{k} y_i^2\right)^{1/2} < \frac{\delta}{2k}$

If $y(t)$ - sol. $\dot{y} = By + G(y)$, $y(0) \in \tilde{S}$, $y(0) = u(0, a)$
$\Rightarrow y(t) = u(t, a)$

From the bound (**), if $y(t)$ - sol.
and. $y(0) \in \tilde{S} \Rightarrow y(t) \to 0$ as $t \to \infty$.
$y(0) \notin \tilde{S} \Rightarrow y(t) \not\to 0$ as $t \to \infty$.

Can show that $\frac{\partial \Psi_j}{\partial y_i}(0) = 0 \quad \begin{matrix} i = 1, \dots k \\ j = k+1, \dots n \end{matrix}$

$\Rightarrow \tilde{S}$ is tangent to the stable subspace E^s
$E^s = \{y_1 = \dots = y_k = 0\}$ for $\dot{y} = By$ at 0.

Same can be done to establish existence of $\tilde{\mathcal{U}}$ - unstable manifold in y-space
(have to consider $\dot{y} = -By - G(y)$, $t \to -t$).

In the x space, $S = C \cdot \tilde{S}$ - stable manifold
$\mathcal{U} = C \cdot \tilde{\mathcal{U}}$ - unstable

<u>Ex. 1</u>. $\begin{cases} \dot{x}_1 = -x_1 - x_2^2 \\ \dot{x}_2 = x_2 + x_1^2 \end{cases}$

Find $S, \mathcal{U}$ by successive approximations

$S: x_2 = \Psi_2(x_1)$

$$F(x) = G(x) = \begin{pmatrix} -x_2^2 \\ x_1^2 \end{pmatrix} \qquad A = B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix}, \quad V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{t} \end{bmatrix} \qquad e^{At} = U(t) + V(t)$$

$$a = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} \qquad u(t,a) = \begin{bmatrix} u_1(t,a) \\ u_2(t,a) \end{bmatrix}$$

$$u(t,a) = \begin{bmatrix} e^{-t} a_1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-s)} u_2^2(s,a) \\ 0 \end{bmatrix} ds - \int_t^\infty \begin{bmatrix} 0 \\ e^{t-s} u_1^2(s,a) \end{bmatrix} ds$$

$$u^{(0)}(t,a) = 0.$$

$$u^{(1)}(t,a) = \begin{bmatrix} e^{-t} a_1 \\ 0 \end{bmatrix}$$

$$u^{(2)}(t,a) = \begin{bmatrix} e^{-t} a_1 \\ 0 \end{bmatrix} - \int_t^\infty \begin{bmatrix} 0 \\ e^{t-s} e^{-2s} a_1^2 \end{bmatrix} ds = \begin{bmatrix} e^{-t} a_1 \\ -\frac{e^{-2t}}{3} a_1^2 \end{bmatrix}$$

$$u^{(3)}(t,a) = \begin{bmatrix} e^{-t} a_1 \\ 0 \end{bmatrix} - \frac{1}{9} \int_0^t \begin{bmatrix} e^{-(t-s)} e^{-4s} a_1^4 \\ 0 \end{bmatrix} ds - \int_t^\infty \begin{bmatrix} 0 \\ e^{t-s} e^{-2s} a_1^2 \end{bmatrix} ds$$

$$= \begin{pmatrix} e^{-t} a_1 + \frac{1}{27}(e^{-4t} - e^{-t}) a_1^4 \\ -\frac{1}{3} e^{-2t} a_1^2 \end{pmatrix}$$

$$u^{(4)}(t,a) - u^{(3)}(t,a) = O(a_1^5)$$

$\Psi_2(a_1) = u_2(0, a_1, 0)$ is approximated by

$$\Psi_2(a_1) = -\tfrac{1}{3} a_1^2 + O(a_1^5)$$

$$S = \{(x_1, x_2) : x_2 = -\tfrac{1}{3} x_1^2 + O(x_1^5)\}$$ Stable manifold as $x_1 \to 0$.

$C = I$, so no transformation to x-space needed.

By choosing $t \to -t$: $U = \{x_1 = -\frac{x_2^2}{3} + O(x_2^5)\}$ $x_2 \to 0$.