Math 677.
Lecture 11.

Ex. Predator - prey model: $x(t)$ - prey, $y(t)$ - predators

$$\begin{cases} \dfrac{dx}{dt} = \gamma x\left(1-\dfrac{x}{K}\right) - c\,\dfrac{mx}{a+x}\,y \\ \dfrac{dy}{dt} = \left(\dfrac{mx}{a+x} - d\right)y \end{cases}$$

$x(0) > 0$
$y(0) > 0$

Let $m > d$.

1) grows logistically
2) d - death rate of predators
3) c - ratio of reproduction of predator over consumption of prey
4) γ - growth rate of prey in absence of predator.

Equilibria: $(0,0)$ $(K,0)$ (x^*, y^*)

$$x^* = \frac{a}{\frac{m}{d} - 1} > 0 \qquad y^* > 0 \text{ if } x^* < K.$$

Linearization:

$$Df(x,y) = \begin{bmatrix} \gamma\left(1-\dfrac{2x}{K}\right) - \dfrac{ma}{(a+x)^2}y & -\dfrac{mx}{a+x} \\ \dfrac{ma}{(a+x)^2}y & \dfrac{mx}{a+x} - d \end{bmatrix}$$

1) $P = (0,0)$ $\quad Df(0,0) = \begin{pmatrix} \gamma & 0 \\ 0 & -d \end{pmatrix} \Rightarrow$ saddle for $\gamma > 0, d > 0$.

2) $P = (K,0)$ $\quad Df(K,0) = \begin{pmatrix} -\gamma & -\dfrac{mK}{a+K} \\ 0 & \dfrac{mK}{a+K} - d \end{pmatrix}$ - saddle for $x^* < K$.

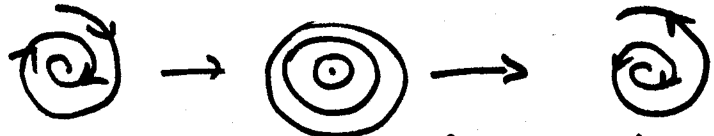
3) $P = (x^*, y^*)$ $\quad Df(x^*,y^*) = \begin{bmatrix} \gamma\left(1-\dfrac{2x^*}{K}\right) - \dfrac{ma}{(a+x^*)^2}y^* & -\dfrac{mx^*}{a+x^*} \\ \dfrac{ma}{(a+x^*)^2}y^* & 0 \end{bmatrix}$

$\lambda^2 - a_1\lambda + a_2 = 0$
$a_1 = \lambda_1 + \lambda_2, \; a_2 = \lambda_1 \cdot \lambda_2$

(a) if $\frac{K-a}{2} < x^*$ $\Rightarrow$ asymp. stable spiral

(b) if $\frac{K-a}{2} > x^*$ $\Rightarrow$ (asymp) unstable spiral

(c) if $\underline{\frac{K-a}{2} = x^*}$ $\Rightarrow$ $\lambda_i = \pm wi$, $w \neq 0$ center



as. stable   stable   unstable

<u>Hopf Bifurcation</u>.

$x^* = \frac{a}{\frac{m}{d} - 1} \gtrless \frac{K-a}{2}$

Values of a, m, d, K giving $\frac{K-a}{2} = x^*$ are called a Bifurcation point

<u>Stable Manifolds</u>.

We will show:

1) E^s, E^u – stable, unstable subsets for $\dot{x} = Ax$, $A = Df(x_0)$

$\Rightarrow$ S, U – stable, unstable manifolds for $\dot{x} = f(x)$ exist and are tangent to E^s, E^u at x_0.

2), S, U - have same dim. as E^s, E^u

$\lim_{t\to\infty} \varphi_t(c) = x_0$, $c \in S$, $\lim_{t\to-\infty} \varphi_t(c) = x_0$, $c \in U$.

Assume $x_0 = 0$ ($y = x - x_0$ $\leftarrow$ transformation to a system with $(0,0)$ as an equilibrium).

<u>Ex</u>. $\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 + x_1^2 \\ \dot{x}_3 = x_3 + x_1^2 \end{cases}$ $\quad x(0) = \vec{c}$

$A = Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E^s = \{x_1 x_2\text{-plane}\}$

$E^u = \{x_3\text{-axis}\}$

$\begin{cases} x_1 = c_1 e^{-t} \\ x_2 = c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ x_3 = c_3 e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}) \end{cases}$

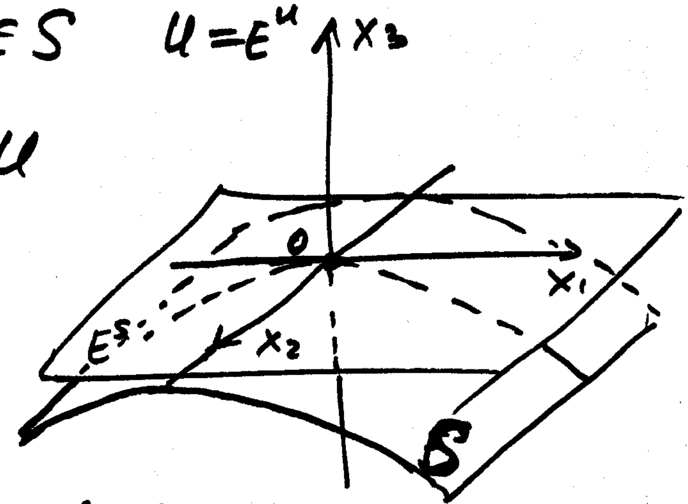
$S = \{c \in \mathbb{R}^3 / c_3 + \frac{c_1^2}{3} = 0\}$

$U = \{c \in \mathbb{R}^3 / c_1 = c_2 = 0\}$

$\lim_{t\to\infty} \varphi_t(c) = 0$ if $c \in S$

$\lim_{t\to-\infty} \varphi_t(c) = 0$ if $c \in U$

$S = \{x \in \mathbb{R}^3 \mid x_3 = -\frac{x_1^2}{3}\}$



Def. X - metric, $A, B \subset X$ subsets

$h : A \to B$ s.t.

(1) h - continuous

(2) h maps A onto B.

(3) $h^{-1} : B \to A$ - continuous

$\Rightarrow h$ - is a homeomorphism $\quad (A \underset{h}{\cong} B)$

Def. M - n-dim differential manifold of C^k-type

if (1) M - connected metric space

(2) $M = \bigcup_\alpha U_\alpha$, U_α - open subsets $\leftarrow$ open covering of M

(3) $\forall \alpha \quad U_\alpha \cong B = \{x \in \mathbb{R}^n \mid |x| < 1\}$

$h_\alpha : U_\alpha \to B$ - homeo

(4) if $U_\alpha \cap U_\beta \neq \phi$, $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ - homeo

$\Rightarrow h_\alpha(U_\alpha \cap U_\beta), h_\beta(U_\alpha \cap U_\beta)$ - subsets of $\mathbb{R}^n$

and $h = h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$ is C^k-differentiable $\forall x \in h_\beta(U_\alpha \cap U_\beta)$

$|Dh(x)| \neq 0$

M - analytic if $h = h_\alpha \circ h_\beta^{-1}$ is analytic for all α & β.

(U_α, h_α) - charts

$U(U_\alpha, h_\alpha)$ - atlas of M.

If $\exists$ atlas s.t. $|Dh_\alpha \circ h_\beta^{-1}(x)| > 0$ for $\forall \alpha, \beta$ $\forall x \in h_\beta(U_\alpha \cap U_\beta)$

the M - orientable manifold.

Now: ready to prove Stable Manifold Thm.

Remarks:

1) $f \in C^1(E)$, $f(0) = 0 \Rightarrow \dot{x} = f(x)$ can be written in the form $\dot{x} = Ax + F(x)$, $A = Df(0)$ - Jacobian

$F(x) = f(x) - Ax \leftarrow$ higher order terms in Taylor expansion

$F \in C^1(E)$
$F(0) = 0$, $Df(0) = 0$.

2) $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\begin{array}{l}\|x\| < \delta \\ \|y\| < \delta\end{array} \Rightarrow |F(x) - F(y)| < \varepsilon |x - y|$

F - Lipschitz

3) $A = CBC^{-1}$, $|C| \neq 0$ s.t.

$$B = C^{-1}AC = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

P - eigenvalues $\lambda_1, \dots \lambda_k$ s.t. $Re(\lambda_i) < 0$ $i = 1, \dots k$

Q - eigenvalues $Re(\lambda_i) > 0$ $i = k+1, \dots n$

4) Choose $\alpha > 0$ s.t.

$Re(\lambda_i) < -\alpha < 0$ $\quad i = 1, \dots k$

5) $y = C^{-1}x \Rightarrow \dot{y} = By + G(y)$, where

$B = C^{-1}AC$

$G(y) = C^{-1}F(Cy) \in C^1(\tilde{E})$, $\tilde{E} = C^{-1}(E)$.

$\forall \varepsilon > 0$ $\exists \delta > 0$: $|G(y_1) - G(y_2)| \leq \varepsilon |y_1 - y_2|$ $\forall \begin{array}{l}|y_1| \leq \delta \\ |y_2| \leq \delta\end{array}$ Lipschitz condition

We will show:

$\exists\, \psi_j (y_1, \dots y_k)$ differentiable $j = k+1, \dots n$

s.t. $y_j = \psi_j (y_1, \dots y_k)$ defines a differentiable k-dim manifold $\tilde{S}$ in y-space

s.t. S given by $C\tilde{S}$ is the stable manifold of $\dot{x} = f(x)$ in x-space.

Thm. [Stable Manifold Thm]

$E \subset \mathbb{R}^n$-open, $\{0\} \in E$.

$f \in C^1(E)$, φ_t - flow of $\dot{x} = f(x)$, $f(0) = 0$

$Df(0)$ has $\lambda_1, \dots \lambda_n$ - eig. s.t. $Re(\lambda_i) < 0 \; i = 1, \dots k$

$Re(\lambda_i) > 0 \; i = k+1, \dots n$

$\Rightarrow \exists\, k$-dim diff. manifold S, tangent to E^s (stable subset for $\dot{x} = Df(0)x$) at $x_0 = 0$

s.t. 1) $\forall t \geq 0 \quad \varphi_t(S) \subset S$

2) $\forall x_0 \in S \quad \lim_{t \to \infty} \varphi_t(x_0) = 0$ | stable manifold

$\exists\, (n-k)$-dim. manifold U, tangent to E^u

s.t. 1) $\forall t \leq 0 \quad \varphi_t(U) \subset U$

2) $\forall x_0 \in U \quad \lim_{t \to -\infty} \varphi_t(x_0) = 0.$ | unstable manifold

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