Math 677.
Lecture 10.

Def. $E \subset \mathbb{R}^n$ – open $f \in C^1(E)$

$\varphi_t : E \to E$ flow of $\dot{x} = f(x)$ $\forall t \in \mathbb{R}$

Then $S \subset E$ is called invariant under flow φ_t if $\varphi_t(S) \subset S$ for all t.

If $\varphi_t(S) \subset S$ for all $t \geq 0 \Rightarrow$ positively invariant.

$\varphi_t(S) \subset S$ for $\forall t \leq 0 \Rightarrow$ negatively invariant.

In linear case, we had E^s, E^u, E^c – stable, unstable, center manifolds – all invariant under the flow φ_t.

True in general.

Ex. $f(x) = \begin{pmatrix} -x_1 \\ x_2 + x_1^2 \end{pmatrix}$ $\quad x(0) = c \quad \dot{x} = f(x)$

$$\varphi_t(c) = \varphi(t, c) = \begin{bmatrix} c_1 e^{-t} \\ \underbrace{c_2 e^t + \frac{c_1^2}{3}(e^t - e^{-2t})}_{= \left(c_2 + \frac{c_1^2}{3}\right)e^t - \frac{c_1^2}{3}e^{-2t}} \end{bmatrix}$$

$E^s = \{c \in \mathbb{R}^2 / c_2 + \frac{c_1^2}{3} = 0\}$

$E^u = \{c \in \mathbb{R}^2 / c_1 = 0\}$

$= \{x_2\text{-axis}\}$

Suppose $c \in E^s = \{c_2 = -\frac{c_1^2}{3}\}$

$$\varphi_t(c) = \begin{Bmatrix} c_1 e^{-t} \\ -\frac{c_1^2}{3} e^{-2t} \end{Bmatrix} \in E^s \text{ since } \frac{(c_1 e^{-t})^2}{3} = -c_2$$

$-\frac{c_1^2}{3} = c_2$

We want to be able to classify equilibria of any nonlinear system, i.e. analyze its stability.

2 methods: 1) linearization
2) ~~to~~ Lyapunov stability theory

Linearization.

$\dot{x} = f(x)$ (1) $\rightsquigarrow$ $\dot{x} = D_f(x_0)x$

autonomous system

Jacobian at x_0.

x_0 - equilibrium if $f(x_0)=0$.
critical point

<u>Def</u>. x_0 - <u>hyperbolic</u> equilibrium, if $\mathrm{Re}\,\lambda_i(D_f(x_0)) \neq 0$ for all i (all eigenvalues have nonzero real part).

x_0 - <u>sink</u> if $\mathrm{Re}\,\lambda_i(D_f(x_0)) < 0$

<u>source</u> if $\mathrm{Re}\,\lambda_i(D_f(x_0)) > 0$

<u>saddle</u> if $\lambda_i(D_f(x_0))$ have both positive & negative Re parts

Let $y = x - x_0$ $\Rightarrow$ $\dot{y} = D_f(x_0)y$ is the <u>linearization</u> of (1) at x_0, has $y_0 = 0$ as its equilibrium.

If x_0 - equilibrium of (1), φ_t - flow of (1) then $\varphi_t(x_0) = x_0$ so x_0 - fixed point of φ_t.

(1) $\dot{x} = f(x)$ (2) $\dot{x} = D_f(x_0)x$

$$f(x) = D_f(x_0)x + \frac{1}{2} D^2 f(x_0) x^2 + \dots$$

for x - close to x_0 by Taylor.

Are the solutions close? in terms of stability, the answer is yes, but not always.

Stability of (1) is nonlinear stability
of (2) is linearized stability.

<u>Claim</u>: Linearized stability implies nonlinear stability at x_0 for any hyperbolic equilibrium x_0.

<u>Ex. 1</u> $\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2) \end{cases}$ $(0,0)$ - equilibrium

$$A = D_f(0,0) = \left.\begin{bmatrix} -(x_1^2 + x_2^2) - 2x_1^2 & 1 - 2x_1x_2 \\ -1 - 2x_1x_2 & -(x_1^2 + x_2^2) - 2x_2^2 \end{bmatrix}\right|_{(0,0)} =$$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\dot{y} = Ay$ $\lambda_{12} = \pm i\omega$
$(0,0)$ - center (stable)

$$\dot{x}_1 \cdot x_1 + \dot{x}_2 \cdot x_2 = -(x_1^2 + x_2^2)^2$$

$r^2(t) = x_1^2 + x_2^2$ (radius-vector) $\Rightarrow \frac{1}{2}\frac{d}{dt} r^2(t) = -(r^4(t))$

$\Rightarrow r^2(t) = \frac{c}{1+2ct}$, $c = x_1^2(0) + x_2^2(0)$ $\to 0$ as $t \to \infty \Rightarrow$ asymp. stability

Ex. 2 $\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$ system (1)

(0,0)

$A x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$ (0,0) - center for linearized (1) system

(stable in first approximation)

$$\frac{d}{dt} r^2(t) = 2\left(r^2(t)\right)^2$$

$$r^2(t) = \frac{r^2(0)}{1 - 2r^2(0) \cdot t}$$

$r(t) \to \infty$ in finite time (solution blows up in finite time).

as $t \to t^* > 0$

__Want to show:__ this does not happen for hyperbolic critical pts.

__Ex. 3__ $\dot{x} = f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$ (1) $\quad \dot{x} = Ax \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (2)

Consider $H(x) = \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} \end{bmatrix}$ - cts map from $\mathbb{R}^2$ onto $\mathbb{R}^2$

$$H^{-1}(y) = \begin{bmatrix} y_1 \\ y_2 - \frac{y_1^2}{3} \end{bmatrix}$$

$$H^{-1}(H(x)) = H^{-1}\begin{pmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} - \frac{x_1^2}{3} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

H transforms (1) into (2).

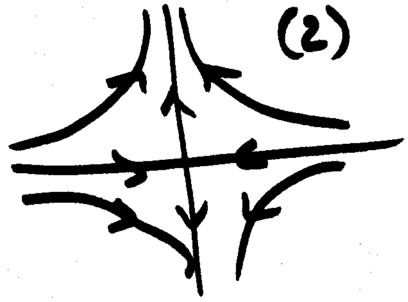
$$y = Hx \Rightarrow \dot{y} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 + \frac{2\dot{x}_1 \cdot x_1}{3} \end{bmatrix} = \begin{matrix} -x_1 \\ x_2 + \cancel{\ldots} \end{matrix}$$

$$\| \quad \begin{bmatrix} x_1 \\ x_2 + \frac{x_1^2}{3} \end{bmatrix}$$

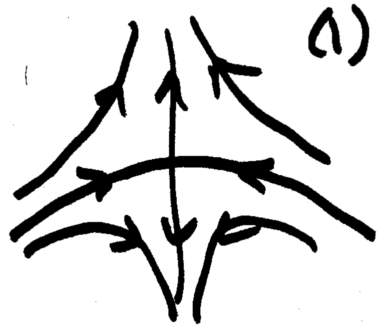
$$\dot y = \begin{pmatrix} \dot x_1 \\ \dot x_2 + \frac{2}{3} x_1 \cdot \dot x_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 + x_1^2 + \frac{2}{3} x_1(-x_1) \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 + \frac{1}{3} x_1^2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix}$$

$$\dot y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} y \leftarrow$$

(2)



(1)



<u>Recall</u>:

$$\begin{cases} \dot x = A(t)x + g(t) \\ x(\tau) = \xi \end{cases} \leftarrow \varphi(t,\tau,\xi) \text{ - solution}$$

$\dot x = A(t)x \quad \leftarrow \Phi(t)$ - <u>fundam. matrix</u> of solutions

$= [\varphi_1, \ldots \varphi_n]$

φ_i - solution of $\dot x = Ax$

$\{\varphi_i\}$ - lin. indep.

Variation of constants formula:

$$\boxed{\varphi(t,\tau,\xi) = \Phi(t)\Phi^{-1}(\tau)\xi + \int_\tau^t \Phi(t)\Phi^{-1}(\eta)g(\eta)\,d\eta}$$

<u>Thm.</u> (Linearized Stability Principle)

x^0 - equilibrium of autonomous system $\dot x = f(x)$, $\operatorname{Re} \lambda_i(D_f(x_0)) < 0$ for all eigenvalues $\lambda_i \Rightarrow x_0$ - asymp. stable equilibrium for $\dot x = f(x)$.

Proof. $A = D_f(x)$ $y = x - x^0$

$y' = f(x) = f(y + x^0) = \underbrace{Ay}_{\text{linearization}} + g(y)$ — involves higher powers of y starting with y^2

$g(y) = o(|y|)$ as $y \to 0$ $g(0) = 0$.

Since $\operatorname{Re}\lambda(A) < 0 \Rightarrow$

from linear stability we know that $\exists M, \sigma > 0$ s.t. $|e^{At}| \le M e^{-\sigma t}, t \ge 0$

Variation of constants formula:

$$y(t) = e^{A(t-t_0)} y(t_0) + \int_{t_0}^{t} e^{A(t-s)} g(y(s))\, ds$$

$$|y(t)| \le M |y(t_0)| \cdot e^{-\sigma(t-t_0)} + M \int_{t_0}^{t} e^{-\sigma(t-s)} |g(y(s))|\, ds$$

Since $g(y) = o(|y|)$ $\forall\, 0 < \varepsilon < \sigma$ $\exists \delta > 0$ s.t.

$|g(y)| < \frac{\varepsilon}{M} |y|$ for all $|y| < \delta$ $\oplus$

Take $0 < d < \min\left(\frac{\delta}{2M}, \frac{\delta}{2}\right)$

Claim: if $|y(t_0)| < d \Rightarrow |y(t)| < \frac{\delta}{2}$

$|y(t)| \le \frac{\delta}{2} e^{-(\sigma-\varepsilon)(t-t_0)}$ $\forall t \ge t_0$

Suppose $|y(t)| < \frac{\delta}{2}$ for $t_0 \le t < t^*$, $y(t^*) = \frac{\delta}{2}$

$\Rightarrow$ from $\oplus$ $|y(t)| \le M |y(t_0)| \cdot e^{-\sigma(t-t_0)} + \varepsilon \int_{t_0}^{t} e^{-\sigma(t-s)} |y(s)|\, ds$

Apply Gronwall lemma to $e^{\sigma t} y(t)$:

$\Rightarrow |y(t)| \le M |y(t_0)| e^{-(\sigma-\varepsilon)(t-t_0)} < \frac{\delta}{2} e^{-(\sigma-\varepsilon)(t-t_0)}$

$\Rightarrow$ $y(t)$ - asymp. stable at ~~x~~ $(0,0)$.
(exponentially stable) $\blacksquare$