§1. Linear systems.

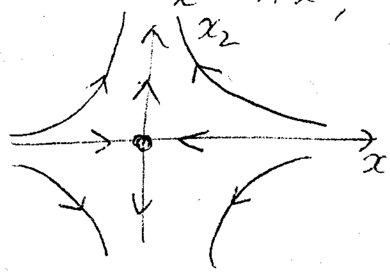
$$\begin{cases}\dot{\vec{x}} = A\vec{x} \\ \vec{x}(0) = \vec{x}_0\end{cases} \Rightarrow \quad \vec{x}(t) = e^{At}\vec{x}_0$$

(1.1.) Uncoupled systems

$$\dot{x} = ax \Rightarrow x(t) = ce^{at}$$

Ex. 1
$$\begin{cases}\dot{x}_1 = -x_1 \\ \dot{x}_2 = 2x_2\end{cases} \Rightarrow \quad \begin{matrix} x_1 = C_1 e^{-t} \\ x_2 = C_2 e^{2t}\end{matrix} \Rightarrow \vec{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t}\end{bmatrix} c$$

$\dot{x} = Ax$, $A = \begin{pmatrix} -1 & 0 \\ 0 & 2\end{pmatrix}$ diagonal



(x_1, x_2) · phase plane

$(0,0)$ - equilibrium $\lambda_1 = -1$, $\lambda_2 = 2$ saddle

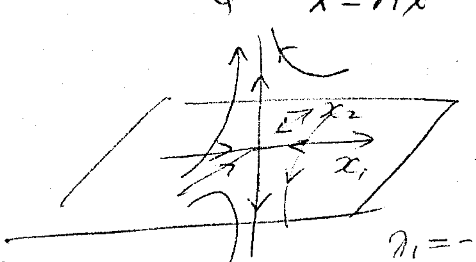
Dynamical system: $\varphi: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi(t, \vec{c}) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t}\end{bmatrix}\vec{c}$$

$\dot{x} = f(x)$: defines a vector field $f: \mathbb{R}^2 \to \mathbb{R}^2$ (tangent or slope field since solution curves are always tangent to v.f. Ax; $\dot{x}_0(t_0) = v_0$ tangent to $x = x(t)$ at $x_0 = x(t_0)$)

φ $\dot{x} = Ax$

Ex. 2
$$\begin{cases}\dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \\ \dot{x}_3 = 2x_3\end{cases}$$



$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2\end{pmatrix}$$

$\lambda_1 = -1$ $A - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\end{pmatrix}$ $x_3 = 0$

$\lambda_3 = 2$ $A - \lambda I = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0\end{pmatrix}$ $x_1 = x_2 = 0$

(x_1, x_2) - stable subspace

x_3 - axis - unstable subspace

In general, $\lambda_1, \ldots \lambda_k < 0$ $v_1, \ldots v_k$ - eig. v

$\lambda_{k+1}, \ldots \lambda_n > 0$ (distinct) $v_{k+1}, \ldots v_n$ - eig. v.

$\Rightarrow E^s = \text{Span}\{v_1, \ldots v_k\}$ = stable subspace

$E^u = \text{span}\{v_{k+1}, \ldots v_n\}$ = unstable subspace

1.2 Diagonalization

Thm. $\lambda_1, \lambda_2 \ldots \lambda_n$ - real, distinct
$v_1, \ldots v_n$
$\Rightarrow \{v_1, \ldots v_n\}$ forms a basis in $\mathbb{R}^n$ s.t. $P = [v_1 \ldots v_n]$ is invertible and $P^{-1}AP = \operatorname{diag}[\lambda_1, \ldots \lambda_n]$.

$$A = P\Lambda P^{-1}$$

Corollary: $\dot{x} = Ax$, A - diagonalizable $P^{-1}AP = \operatorname{diag}(\lambda_1 \ldots \lambda_n)$

$$\Rightarrow \quad x(t) = PE(t)P^{-1}x(0), \quad E(t) = \operatorname{diag}[e^{\lambda_1 t}, \ldots e^{\lambda_n t}]$$

Proof. $y = P^{-1}x \Rightarrow x = Py$

$$A = P^{+1}\Lambda P^{-1}$$
$$\dot{x} = Ax = P^{+1}\Lambda P^{-1}x$$
$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy = \Lambda y \quad \text{(uncoupled)}.$$
$$\Rightarrow y = e^{\Lambda}y_0 = \underbrace{\operatorname{diag}(e^{\lambda_1 t}, \ldots e^{\lambda_n t})}_{= E(t)} y_0$$

$$P^{-1}x = y = E(t)y_0 = E(t)P^{-1}x(0)$$
$$\Rightarrow x = PE(t)P^{-1}x(0)$$

Ex. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}$ $\quad \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 2 \\ \lambda_3 = 1 \end{matrix}$ $\quad v_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ $\quad P = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

$$E(t) = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

$\dot{x} = Ax$

$x = PE(t)P^{-1}x(0)$

$$= \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \underbrace{P^{-1}}_{= \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right] \sim$$

$y = P^{-1}x \quad \begin{cases} \dot{y}_1 = 3y_1 & y_1 = C_1 e^{3t} \\ \dot{y}_2 = 2y_2 & y_2 = C_2 e^{2t} \\ \dot{y}_3 = y_3 & y_3 = C_3 e^{t} \end{cases}$

$$X = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix} \begin{bmatrix} 0 & -e^{3t} & 0 \\ 2e^{2t} & 0 & e^{2t} \\ -e^{t} & e^{t} & 0 \end{bmatrix} = \begin{bmatrix} e^{t} & e^{3t}-e^{t} & 0 \\ 0 & e^{3t} & 0 \\ 2e^{2t}-2e^{t} & -2e^{3t}+2e^{t} & e^{2t} \end{bmatrix} X_0$$

$$x_1(t) = c_1 e^{t} + c_2(e^{3t} - e^{t}) \qquad X_0 = [c_1,\ c_2\ \ c_3]$$

$$x_2(t) = c_2 e^{3t}$$

$$x_3(t) = c_1(2e^{2t} - 2e^{t}) + c_2(-2e^{3t} + 2e^{t}) + c_3 e^{2t}$$

Ex. $\ddot{x} + 3\dot{x} + 2x = 0$

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 3x_2 \end{array}\right. \qquad \dot{\vec{x}} = A\vec{x} \quad A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda(3+\lambda) + 2 = \lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0 \qquad \lambda_1 = -1,\ \lambda_2 = -2$$

$$\lambda_1 = -1 \quad \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1^1 + v_1^2 = 0 \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = -2 \quad \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \Rightarrow 2v_2^1 + v_2^2 = 0 \quad V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad P^{-1} = -\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{vmatrix} = \frac{1}{|A|}\cdot(\mathrm{adj}(A))^{T}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$X = PE(t)P^{-1}x(0) = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} x(0)$$