Math 677. Fall 2009.
Homework #7 Solutions.

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 2.13: # 1
The system

$$\dot{x} = y + ax^2 + bxy + cy^2$$
$$\dot{y} = dx^2 + exy + fy^2$$

Let $L_J[h(x)] = Jh(x) - Dh(x)J(x)$ and $h_2 = [a_{20}x^2 + a_{11}xy + a_{02}y^2; b_{20}x^2 + b_{11}xy + b_{02}y^2]$. Then $L_J[h_2(x)] = [0; dx^2 + (e + 2a)xy]$.

Problem Set 2.13: # 5
The system

$$\dot{x} = y + x^2 - x^3 + xy^2 - y^3$$
$$\dot{y} = x^3 - 2xy + x^3 - x^2y$$

can be reduced to normal form by the following sequence of steps. First consider change of variables $x = z + (0, -z_1^2)$ and let $\dot{x} = Jx + F_2(x) + F_3(x)$, where F_2 consists of 2nd order terms. Substituting z into the equation and expressing everything in terms of the new variables, we get

$$\dot{z}_1 = z_2 + (-z_1^3 + z_1z_2^2 - z_2^3) + O(|z|^4)$$
$$\dot{z}_2 = z_1^2 + (3z_1^3 - z_1^2z_2) + O(|z|^4)$$

Next, introduce another change of variables in the form $z = u + (0, u_1^3 - u_1u_2^2 + u_2^3)$. After some algebra this yields

$$\dot{u}_1 = u_2 + O(|u|^4)$$
$$\dot{u}_2 = u_1^2 + 3u_1^3 - 4u_1^2u_2 + O(|u|^4)$$

which defines the normal form for this system.

Problem Set 2.14: # 2
The Hamiltonian of the system

$$\dot{x} = y$$
$$\dot{y} = -x + x^2$$

is given by $H(x, y) = y^2/2 + x^2/2 - x^3/3$. According to Theorem 3, there is a center at the origin (strict local min) and saddle at $(1, 0)$ (strict local max).

Problem Set 4.3: # 2

Write the system as

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} xy \\ -x^2 \end{bmatrix}$$

Due to the zero eigenvalue, the behavior near the origin is completely determined by the flow on the center manifold. We see that $C = 0, P = [-1], F(x, y) = xy, G(x, y) = -x^2$. So $h'(x)[Cx + F(x, h(x))] - Ph(x) - G(x, h(x)) = 0$ yields $h(x) = ax^2 + O(x^3)$ with $a = -1$. Hence the flow at the center manifold is given by $\dot{x} = -x^3$. Universal unfolding is then given by

$$\begin{bmatrix} \dot{x} = \mu_1 + \mu_2 x - x^3 \\ \dot{y} = -y \end{bmatrix}$$

similar to the pitchfork bifurcation. The nonlinear terms in the secodn equation can be removed by writing the normal form of the system.

Part II.

The "6-12" potential is given by $U(x) = \frac{a}{12x^{12}} - \frac{b}{6x^2}$ corresponds to a Newtonian system with $f = -U'(x) = \frac{a}{x^{13}} - \frac{b}{3x^3}$. Hamiltonian is provided by $H(x, y) = \frac{y^2}{2} + \frac{a}{12x^{12}} - \frac{b}{6x^2}$. The unique critical point is $((3a/b)^{1/10}, 0)$ (the negative one does not have a physical meaning). The point corresponds to a center of the Newtonian system.