Math 677. Fall 2009.
Homework #6 Solutions.

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 2.11: # 2

(a) $\dot{x} = x^2, \dot{y} = y$. Expansion of the function $\psi = p_2(x, \phi)$ is a nbhd of the origin has the form $\psi = a_m x^m + \ldots$ with $m \geq 2, a_m \neq 0$. $m = 2$, so the origin is a saddle node.
(b) $\dot{x} = x^2 + y^2, \dot{y} = y - y^2$. Same as above.
(c) $\dot{x} = y^2 + x^3, \dot{y} = y - x^2 . m = 3, a_m = 1$, so unstable node.

Problem Set 2.12: # 1

$\dot{x} = x^2, \dot{y} = -y$. Linearization gives $\lambda = 0, -1$, so $c = 1, s = 1, n = 2$. Take $h(x) = ax^2 + bx^3 + \ldots$. Plug into $Dh(x)[Cx + F(x, h(x)] - Ph(x) - G(x, h(x)) = 0$, we get $h(x) = 0, \forall x$. $W^c(0) = E^c$. By direct computation,

$$h(x, c) = \begin{cases} 0, & \text{for} \quad x \geq 0 \\ ce^{1/x}, & \text{for} \quad x < 0 \end{cases}$$

satisfies the above equation, so it defines a C^∞ center manifold for the system.

Problem Set 2.12: # 2

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y + ax^2 + xy \end{aligned}$$

The eigenvalues of the linearlized system are $\lambda = 0, -1$. The Jordan form is

$$J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

So we have $C = 0, P = [-1], F(x, y) = 0, G(x, y) = ax^2 + xy$. Expand $h(x) = ax^2 + \ldots$ in the equation for center manifold to get the system's flow on the center manifold as $\dot{x} = x^2 + O(|x|^3)$.

Part II, #1

$$\begin{aligned} \dot{x} &= -xy \\ \dot{y} &= -y + x^2 - 2y^2 \end{aligned}$$

We get $\lambda = 0, -1$, $C = 0, P = [-1], F(x,y) = -xy, G(x,y) = x^2 - 2y^2$. Let $h(x) = ax^2 + O(x^3)$. Then by the Center Manifold Theorem, $-4a^2x^3 + ax^2 - x^2 + 2y^2 = 0$, which means that $a = 1$. Therefore $\dot{x} = -x^3 + O(x^4)$.

Part II, #2

(a) $E^s = \{y - axis\}, E^c = \{x_1, x_2 - plane\}$.

(b) By CMT, $h(x) = ax_1^2 + bx_1x_2 + cx_2^2$ should satisfy $a = -1, b = 0, c = -1$, i.e. $C = \{y = h(x) = -x_1^2 - x_2^2 +)(|x|^3)$.

(c) The equation of the flow on the center manifold is then

$$\dot{x}_1 = -x_2 - x_1x_2^2 - x_1^3 + O(|x|^3)$$
$$\dot{x}_2 = x_1 - x_2^3 - x_1^2x_2 + O(|x|^3)$$

(d) In polar coordinates $\dot{r} = -r^3 + (|r|^3), \dot{\theta} = 1 + O(|\theta|^3)$