Math 677. Fall 2009.
Homework #5 Solutions.

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 2.9: # 3
For the system

$$\dot{x} = \begin{cases} -x_2 - x_1x_2^2 + x_3^2 - x_1^3 \\ x_1 + x_3^2 - x_2^3 \\ -x_1x_3 - x_3x_1^2 - x_2x_3^2 - x_3^5 \end{cases}$$

the Liapunov function $V = x_1^2 + x_2^2 + x_3^2$ yields $\dot{V} = -2(x_1^4 + x_2^4 + x_3^6 + x_1^2(x_2^2 + x_3^2)) \leq 0$. Since $V(x) > 0$ with $V = 0$ iff $x = 0$, the function is a strict Liapunov function and hence the origin is an asymptotically stable equilibrium. Linearized system has eigenvalues $\lambda = \pm i$ and $\lambda = 0$, so the trajectories are circles parallel to the $x_1 - x_2$ plane.

Problem Set 2.9: # 7
Lienard system $\ddot{x} + f(x)\dot{x} + g(x) = 0$ can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 - F(x_1) \\ \dot{x}_2 &= -g(x_1) \end{aligned}$$

where $F(x) = \int_0^x f(s)ds$. This can be verified by a direct calculation (taking $\ddot{x}_1$). Now take the Liapunov function in the form $V(x) = G(x_1) + \frac{x_2^2}{2}$ - it is positive since $G(x) > 0$. We get $\dot{V} = -g(x_1)F(x_1)$, so if $g(x)F(x) > 0$ in a deleted nbhd of the origin, the origin is asymptotically stable.

Problem Set 2.10: #1 (c) The system

$$\begin{aligned} \dot{x} &= -y + x^5 \\ \dot{y} &= x + y^5 \end{aligned}$$

in polar coordinates gives $\dot{r} = r^5(\sin^6\theta + \cos^6\theta)$, $\dot{\theta} = 1 - \frac{1}{4}r^4\sin^4\theta$. This defines an unstable focus.

Part II.

$$\begin{aligned} \dot{x} &= y - x^2 \\ \dot{y} &= x - y^2 \end{aligned}$$

(a) $(0,0)$ is a saddle and $(1,1)$ is a stable node for the linearized system.

(b) By Liapunov function $V(x) = \frac{1}{2}((x-1)^2 + (y-1)^2)$, we get $\dot{V} < 0$ for $|x-1| < 1, |y-1| < 1$, so $(1,1)$ is asymp. stable.

To prove that $(0,0)$ is unstable, it is enough to have $V(x)$ s.t. $V > 0$ for at least some x arbitrary close to the critical point s.t. $\dot{V} > 0$. This can be accomplished via the choice of $V = xy$.