Math 677. Fall 2009.
Homework #4 Solutions.

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 2.5: # 5
Determine the flow of the nonlinear system $\dot{x} = f(x), f(x) = (-x_1, 2x_2+x_1^2)^T$ and show that the set $S = \{x_2 = -\frac{1}{4}x_1^2\}$ is invariant with respect to the flow.

Solution to the IVP:

$$x_1(t) = c_1 e^{-t}$$
$$x_2(t) = (c_2 + \frac{1}{4}c_1^2)e^{2t} - \frac{1}{4}c_1^2 e^{-2t}$$

where $x(0) = (c_1, c_2)^T$. The flow is defined by

$$\phi(x,t) = \begin{bmatrix} x_1 e^{-t} \\ (x_2 + \frac{1}{4}x_1^2)e^{2t} - \frac{1}{4}x_1^2 e^{-2t} \end{bmatrix}$$

To check invariance, start with $y \in S$, i.e. $y_2 = -\frac{1}{4}y_1^2$. Then $\phi(y,t) = (y_1 e^{-t}, -\frac{1}{4}y_1^2 e^{-2t})$, so that $\phi(y,t) \in S$ as well.

Problem Set 2.6: # 2
For the Lorenz equation $\dot{x} = f(x), f(x) = (x_2-x_1, \mu x_1-x_2-x_1x_3, x_1x_2-x_3)^T$, $\mu > 0$, the equilibrium point satisfies

$$\vec{x}_0 = (a, a, a^2), \text{ where} \mu a - a - a^3 = 0$$

Obviously, $x_0 = 0$ is one of the equilibria. Jacobian at the origin has eigenvalues $\lambda_1 = -1, \lambda_{2,3} = -1 \pm \sqrt{\mu}$, which is a sink when $\mu < 1$ and a saddle when $\mu > 1$.

Other equilibria satisfy $\mu - 1 = a^2$. It means that when $\mu > 1$, there are two more points bifurcating from the origin at $x_0 = (a, a, a^2)$ with $a = \pm\sqrt{\mu - 1}$.

Jacobian here can be computed as

$$Df = \begin{bmatrix} -1 & 1 & 0 \\ \mu - x_3 & -1 & -x_1 \\ x_2 & x_1 & -1 \end{bmatrix}, \text{ i.e.} Df(x_0) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -a \\ a & a & -1 \end{bmatrix}$$

It follows that $\lambda_1 = -2, \lambda_{2,3} = \frac{1}{2}(-1 \pm \sqrt{5-4\mu})$ are the eigenvalues. Here if $1 < \mu \leq 5/4$, we have $\lambda_{2,3} < 0$ i.e. x_0 is a sink. Likewise for $\mu > 5/4$, we have a focus with negative real part of $\lambda_{2,3}$, which also can be classified as a sink.

Hence $\mu = 1$ is the bifurcation point. When $\mu < 1$, there is a sink at the origin. When $\mu = 1$, the sink becomes degenerate and further for $\mu > 1$ it splits into sinks (a, a, a^2) with $a = \pm\sqrt{\mu - 1}$, forming a saddle point at the location of the original equilibrium $((0,0))$.

Problem Set 2.6: # 3

For $H(x) = (x_1, x_2 + x_1^2, x_3 + \frac{1}{3}x_1^2)^T$, we compute $H^{-1}(x) = (x_1, x_2 - x_1^2, x_3 - \frac{1}{3}x_1^2)^T$. Both maps are clearly continuous. Take the nonlinear system $\dot{x} = f(x)$ with $f(x) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)^T$ and let $y = Hx$. Then

$$\dot{y} = \dot{H}(x) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 + 2x_1\dot{x}_1 \\ \dot{x}_3 + \frac{2}{3}x_1\dot{x}_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 - x_1^2 \\ x_3 + \frac{1}{3}x_1^2 \end{bmatrix} = Df(0)y$$

Hence H is a homeomorphism between the nonlinear system and its linearization at the origin.

Problem set 2.7: #4.

Find the first 4 successive approximations for the system

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= x_3 + x_2^2 \end{aligned}$$

The system has a hyperbolic saddle at the origin, since $Df(0) = \texttt{diag}\{-1, -1, 1\}$. To find successive approximations, we do not need to translate anything, or diagonalize the matrix $A = Df(0)$, so $A = B, F = G, C = I$.

To find the stable manifold, we fix $a = (a_1, a_2, 0)$ and we are going to look for $x_3 = \psi(x_1, x_2)$. We have $U = \mathtt{diag}\{\mathtt{e}^{-\mathtt{t}}, \mathtt{e}^{-\mathtt{t}}, \mathtt{0}\}$ and $V = \mathtt{diag}\{\mathtt{0}, \mathtt{0}, \mathtt{e}^{\mathtt{t}}\}$, $F = G = (0, x_1^2, x_2^2)^T$.

Applying iterative formula

$$u^{(j+1)} = U(t)a + \int_0^t U(t-s)G(u^{(j)}(s,a))ds - \int_t^\infty V(t-s)G(u^{(j)}(s,a))ds$$

with $u^{(0)}(t,a) = 0$, we obtain:

$u^{(1)}(t,a) = (a_1e^{-t}, a_2e^{-t}, 0)^T$
$u^{(2)}(t,a) = (a_1e^{-t}, (a_2 + a_1^2)e^{-t} - a_1^2e^{-2t}, -\frac{1}{3}a_2^2e^{-2t})^T$
$u^{(3)}(t,a) = (a_1e^{-t}, (a_2 + a_1^2)e^{-t} - a_1^2e^{-2t}, -\frac{1}{3}(a_2 + a_1^2)^2e^{-2t} + \frac{1}{2}a_1^2(a_2 + a_1^2)e^{-3t} - \frac{1}{5}a_1^4e^{-4t})^T$

Since the first two components of the solution stabilize, $u^{(3)} = u^{(4)} = \ldots$. Hence we obtain

$$S = \psi_3(x_1, x_2) = \{x_3 + \frac{1}{3}x_2^2 + \frac{1}{6}x_2x_1^2 - \frac{1}{30}x_1^4\}$$

To calculate the unstable manifold U, we put $a = (0, 0, a_3)$, replace t by $-t$ and express $x_1 = \psi_1(x_3), x_2 = \psi_2(x_3)$. This yields $u(t,a) = u^{(1)}(t,a) = (a_3e^{-t}, 0, 0)^T$. It follows that $a_3 = 0$, so that $U = \{x_1 = x_2 = 0\}$.

Part II. (c)

The system $\dot{x}_1 = -x_1^3, \dot{x}_2 = -x_2$ has equilibrium at the origin, with $E^s = \{y$-axis$\}$ and $E^c = \{x-$axis$\}$. Center manifold should be tangent to the x-axis then. Dividing one equation by the other, we get

$$\frac{dx_2}{dx_1} = \frac{x_2}{x_1^3}$$

which results in the solution $x_2 = Ce^{-1/(2x^2)}$. Clearly, patching together different branches of this solution leads to different center manifolds, all of which are tangent to each other and to E^c at the origin.

$$M_{a,b} = \begin{cases} ae^{-1/(2x^2)}, & x > 0 \\ 0, & x = 0 \\ be^{-1/(2x^2)}, & x < 0 \end{cases}$$

defines a two-parameter continuum family of center manifolds.