`Math 677. Fall 2009.`
`Homework #3 Solutions.`

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 1: # 4
IVP $\dot{x} = x^3, x(0) = 2$ has solution in the form $x(t) = \frac{2}{\sqrt{1-8t}}$, which exists for all $t \in (-\infty, \frac{1}{8})$ and $\lim\limits_{t \to \frac{1}{8}^-} x(t) = \infty$.

Problem Set 3: # 2(a)
IVP $\dot{x} = f(x), x(0) = y$ with $f(x) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)^T$ has solution in the form

$$u(t, y) = (y_1 e^{-t}, y_2 e^{-t} + y_1^2(e^{-t} - e^{-2t}), y_2 e^t + \frac{1}{3} y_1^2 (e^t - e^{-2t}))^T.$$

It follows that

$$\Phi(t, y) = \frac{\partial u}{\partial y}(t, y) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 2y_1(e^{-t} - e^{-2t}) & e^{-t} & 0 \\ \frac{2}{3} y_1 (e^t - e^{-2t}) & 0 & e^t \end{bmatrix}$$

$$Df(x) = \begin{bmatrix} -1 & 0 & 0 \\ 2y_1 e^{-t} & -1 & 0 \\ 2y_1 e^{-t} & 0 & 1 \end{bmatrix}$$

By direct calculation, $A(t, y)\Phi(t, y) = \dot{\Phi}(t, y)$ and $\Phi(0, y) = I$.

Problem Set 4: # 2(c)
IVP $\dot{x}_1 = \dfrac{1}{2x_1}, \dot{x}_2 = x_1, x_1(0) = 1, x_2(0) = 1$ can be solved exactly to obtain $x_1(t) = \sqrt{1+t}, x_2(t) = \dfrac{1}{3} + \dfrac{2}{3}(1+t)^{3/2}$. Maximal interval of existence of the solution is $(-1, \infty)$ and $\vec{x} \to (0, \frac{1}{3})$ as $t \to -1^-$.

Problem Set 4: # 5
To show: if $x_1 = \lim_{t \to \beta^-} x(t)$ exists and $x_1 \in E$, then $\beta = \infty$. In addition, $f(x_1) = 0$ and $x(t) = x_1$ is a solution for IVP with $x(0) = x_1$.

Proof: Define

$$u(t) = \begin{cases} x(t), & 0 \le t \le \beta \\ x_1, & t = \beta \end{cases}$$

It is easy to show that $u(t)$ is continuous on $[0, \beta]$ (use $\epsilon - \delta$ argument), which is a compact set. Consider K to be the image of $u(t)$ for $t \in [0, \beta]$. Clearly $K \subset E$ is compact as an image of compact set under continuous mapping, so by Corollary 2 $\beta = \infty$.

By continuity of f, we have $f(x_1) = f(\lim_{t \to \beta^-} x(t)) = \lim_{t \to \infty} f(x(t)) = \lim_{t \to \infty} \dfrac{dx(t)}{dt} = \dfrac{dx_1}{dt} = 0$.

Part II.

(a) The IVP $\dot{y} = \sqrt{|y|}, y(t_0) = 0$ has at least two solutions $y_1(t) = 0$ and

$$y_2(t) = \begin{cases} \frac{1}{4}(t - t_0)^2, & t \ge t_0 \\ -\frac{1}{4}(t - t_0)^2, & t \le t_0 \end{cases}$$

This does not contradict the Existence and Uniqueness theorem since $f = \sqrt{|y|}$ is not Lipschitz. In the perturbations

$$(i)\, y' = \sqrt{|y|} + \epsilon, y(t_0) = 0, \qquad (ii)\, y' = \frac{y^2}{y^2 + \epsilon^2} \sqrt{|y|}, y(t_0) = 0,$$

though, both right-hand-side functions are Lipschitz. Th unique solution of (i) is given by

$$y_2(t) = \begin{cases} \frac{1}{4}(t - t_0 + 2\sqrt{\epsilon})^2 - \epsilon, & t \ge t_0 \\ -\frac{1}{4}(t - t_0 - 2\sqrt{\epsilon})^2 + \epsilon, & t \le t_0 \end{cases}$$

while (ii) has only trivial solution. This demonstrates that starting from an ODE which does not satisfy any uniqueness condition we can get two drastically different families of solutions by making two different smooth perturbations. The ODE with no uniqueness can be considered a branching point in the space of ODEs.

(b) Picard map for the IVP $\dot{x} = f(t, x), x(t_0) = x_0$ is given by $(Ax)(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau$ and the corresponding sequence $x, Ax, A^2x, \ldots$ is the sequence of Picard approximations. Use these approximations to solve the

IVP $\dot{x} = x, x(0) = 1$. In this example, it is easy to compute

$$\begin{gathered}
x_0 = 1; \\
x_1 = Ax_0 = 1 + \int_0^t d\tau = 1 + t; \\
x_2 = 1 + \int_0^t (1 + \tau) d\tau = 1 + t + t^2/2; \\
\dots \\
x_n = A^n x_0 = 1 + t + \ldots + t^n/n!
\end{gathered}$$

so that $\lim_{n\to\infty} A^n x_0 = e^t$.

(c) Contraction is a Lipschitz map with constant $L = 1$. Suppose f is Lipschitz with some constant L in general, then $\forall \epsilon > 0$ take $\delta = \epsilon/L$. It follows that $||y - x|| < \delta$ yields $||f(y) - f(x)|| < L||y - x|| < \epsilon$, so we proved continuity.