Math 677. Fall 2009.
Homework #2 Solutions.

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 6: # 4
The system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

has eigenvalues $\lambda_{1,2} = a_1 \pm ib_1 = -1 \pm i$, $\lambda_{3,4} = a_2 \pm ib_2 = 1 \pm i$. Eigenvectors can be chosen for example as $w_{1,2} = u_{1,2} \pm iv_{1,2}$ with $u_1 = (1,0,0,0)^T, v_1 = (0,-1,0,0)^T, u_2 = (0,0,-2,1)^T, v_2 = (0,0,0,1)^T$, so that

$$P = [v_1 u_1 v_2 u_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution of the IVP is then provided by

$$x(t) = P\texttt{diag}\{e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix}\} P^{-1} x_0$$

where $j = 1, 2$.

Problem Set 8: # 6(h)

$$A = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

has eigenvalue $\lambda = 2$ with algebraic multiplicity $n = 4$ and geometric multiplicity $m = 1$, i.e we have $m = 1$ linearly independent eigenvector in the eigenspace of λ and need to compute $n - m = 3$ additional generalized eigenvectors. Direct computation yields $v = (1,0,0,0)^T$ is the eigenvector and

$w_1 = (0,1,0,0)^T, w_2 = (0,-4,1,0)^T, w_3 = (0,12,-3,1)^T$ are the generalized eigenvectors produced by the procedure $(A - \lambda I)w_i = w_{i-1}$, with $i = 1,2,3$ and $w_0 = v$. In the basis determined by $P = [v, w_1, w_2, w_3]$, the Jordan form of A reduces to

$$J = P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem Set 9: # 3

$$\dot{x} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix} x$$

Eigenvalues are $\lambda_1 = 6, \lambda_{2,3} = \pm 2i$, with eigenvectors $w_1 = (0,0,1)^T$ and $w_{2,3} = u \pm iv = (3,1,-1)^T \pm i(-1,3,0)^T$, for instance. Notice that w_1 spans x_3-axis and u, v span x_1, x_2-plane. Since $\lambda_1 > 0$ and $Re(\lambda_{2,3}) = 0$, $E^u = x_3$-axis, $E^c = x_1, x_2$-plane. The phase portrait is a helix around x_3-axis, diverging away from the x_1, x_2-plane, where it forms concentric circles upon projection.

Problem Set 9: # 6
See Notes, lecture 8.
Problem Set 10: # 2

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

with $x(0) = [1,0]^T$, is a linear non-homogeneous system, which is solved via Variations of Constants formula:

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}b(\tau)d\tau$$

Since

$$e^{At} = \begin{bmatrix} e^t & \frac{e^t - e^{-t}}{2} \\ 0 & e^{-t} \end{bmatrix},$$

$$x(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix} + \begin{bmatrix} e^t & \frac{e^t - e^{-t}}{2} \\ 0 & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-\tau} & \frac{e^{-\tau} - e^{\tau}}{2} \\ 0 & e^{\tau} \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix} d\tau$$