Math 677. Fall 2009.
Homework #1 Solutions.

Part I. Exercises are taken from "Diffential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 2: # 3
Write the following linear DE with const coefficients in the form of the linear system $\dot{x} = Ax$ and solve:

(a) $\ddot{x} + \dot{x} - 2x = 0$.
Substitution $x_1 = x, x_2 = \dot{x}$ yields an equivalent 1st order system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 - x_2 \end{cases} \quad \text{i.e. } \dot{x} = Ax, \text{where } A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 1, \lambda_2 = -2$, eigenvectors $v_1 = [1,\ 1]^T, v_2 = [1,\ -2]^T$. Hence

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \text{ with } P^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}.$$

The solution of the IVP is given by

$$x(t) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1} x(0).$$

Problem Set 3: # 4
If T is a linear transformation on $\mathbf{R}^n$ with $||T - I|| < 1$, prove that T is invertible and that the series $\sum_{k=0}^{\infty} (I - T)^k$ converges absolutely to T^{-1}.
To show that T is invertible, notice that

$$||Tx|| = ||(I + (T - I))x|| \geq ||x|| + ||(T - I)x|| \geq \qquad (1)$$
$$\geq ||x|| - ||T - I|| ||x|| = (1 - ||T - I||)||x|| > 0 \qquad (2)$$

for all $||x|| > 0$. In other words, $||Tx|| = 0$ iff $||x|| = 0$, a criterion of invertibility.
The absolute convergence of the series to T^{-1} follows from geometric series formula for matrices.

Problem Set 4: # 4 Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$, eigenvectors $v_1 = (2, -2, 1), v_2 = (0, 0, 1), v_3 = (0, 1, 0)$.

$$P = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

which leads to $x(t) = e^{At}x(0) = P\mathtt{diag}(\lambda_1, \lambda_2, \lambda_3)P^{-1}x(0)$, i.e.

$$x(t) = \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ \frac{1}{2}(e^t - e^{-t}) & 0 & e^{-t} \end{bmatrix} x(0)$$

Problem Set 5: # 5 Second order equation $\ddot{x} + a\dot{x} + bx = 0$ is equivalent to $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

Eigenvalues are given by $\frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$ and $\lambda_1 \cdot \lambda_2 = b, \lambda_1 + \lambda_2 = -a$. According to the standard classification of equilibria for linear systems, we have the following cases:

(1) If $b < 0$, the origin is a saddle.
(2) If $b > 0, a = 0$, it is a center.
(3) $b > 0, a^2 - 4b \geq 0$ is a stable node for $a < 0$ and unstable node for $a > 0$
(4) $b > 0, a^2 - 4b \leq 0$ is a stable focus for $a < 0$ and unstable focus for $a > 0$.

Part II. Prove the following properties of e^A:

(1) if A is diagonalizable, so is e^A
This follows from the fact that $e^A = e^{P^{-1}\Lambda P} = P^{-1}e^{\Lambda}P$.

(2) if A is symmetric, then e^A is positive definite

If $A^T = A$, there is an orthogonal matrix Q s.t. $A = Q^{-1}\Lambda Q$, so that $(e^A x, x) = (Q^T e^\Lambda Q x, x) = (e^\Lambda Qx, Qx) > 0$ for all $x > 0$.

(3) $\mathtt{det}(e^{At}) = e^{\mathtt{tr}A}$

If $A = P^{-1}\Lambda P$, $\mathtt{det} e^{At} = \Pi e^{\lambda_i t}, i = 1, \ldots n$. By the properties of trace, $e^{\mathtt{tr}A} = e^{\lambda_1 + \ldots + \lambda_n} = \Pi e^{\lambda_i t}$. The statement follows from the observation that $\mathtt{tr} P^{-1}\Lambda P = P^{-1}\mathtt{tr}\Lambda P$ and the fact that $A = P^{-1}JP$, where J is composed of Jordan blocks J_i, each of which satisfies the above relation.

(4) $e^{At} = T^{-1}e^{TAT^{-1}t}T$.

This follows from the calculation

$$T^{-1}e^{TAT^{-1}t}T = T^{-1}\sum_{k=0}^{\infty}\frac{(TAT^{-1}t)^k}{k!} = \sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = e^{At}$$