Solutions to Examples on Partial Derivatives

1. (a) $f(x,y) = 3x + 4y;\quad \dfrac{\partial f}{\partial x} = 3;\quad \dfrac{\partial f}{\partial y} = 4.$

(b) $f(x,y) = xy^3 + x^2y^2;\quad \dfrac{\partial f}{\partial x} = y^3 + 2xy^2;\quad \dfrac{\partial f}{\partial y} = 3xy^2 + 2x^2y.$

(c) $f(x,y) = x^3y + e^x;\quad \dfrac{\partial f}{\partial x} = 3x^2y + e^x;\quad \dfrac{\partial f}{\partial y} = x^3.$

(d) $f(x,y) = xe^{2x+3y};\quad \dfrac{\partial f}{\partial x} = 2xe^{2x+3y} + e^{2x+3y};\quad \dfrac{\partial f}{\partial y} = 3xe^{2x+3y}.$

(e) $f(x,y) = \dfrac{x-y}{x+y}.$

$$\frac{\partial f}{\partial x} = \frac{x+y-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2};$$
$$\frac{\partial f}{\partial y} = \frac{-(x+y)-(x-y)}{(x+y)^2} = -\frac{2x}{(x+y)^2}.$$

(f) $f(x,y) = 2x\sin(x^2y).$

$$\frac{\partial f}{\partial x} = 2x\,.\,\cos(x^2y)\,.\,2xy + 2\sin(x^2y) = 4x^2y\cos(x^2y) + 2\sin(x^2y);$$
$$\frac{\partial f}{\partial y} = 2x\,.\,\cos(x^2y)\,.\,x^2 = 2x^3\cos(x^2y).$$

2. $f(x,y,z) = x\cos z + x^2y^3\mathrm{e}^z.$

$$\frac{\partial f}{\partial x} = \cos z + 2xy^3\mathrm{e}^z,$$
$$\frac{\partial f}{\partial y} = 3x^2y^2\mathrm{e}^z,$$
$$\frac{\partial f}{\partial z} = -x\sin z + x^2y^3\mathrm{e}^z.$$

3. (i) $f(x,y) = x^2 \sin y + y^2 \cos x.$

$f_x = 2x \sin y - y^2 \sin x; \qquad f_y = x^2 \cos y + 2y \cos x.$

$f_{xx} = 2 \sin y - y^2 \cos x; \qquad f_{yy} = -x^2 \sin y + 2 \cos x;$

$f_{xy} = 2x \cos y - 2y \sin x; \qquad f_{yx} = 2x \cos y - 2y \sin x.$

So $f_{xy} = f_{yx}$.

(ii) $f(x,y) = \left(\frac{y}{x}\right) \ln x.$

$f_x = \frac{y}{x} \cdot \frac{1}{x} - \frac{y}{x^2} \ln x = \frac{y}{x^2}(1 - \ln x); \qquad f_y = \frac{1}{x} \ln x.$

$f_{xx} = \frac{y}{x^2} \cdot \left(-\frac{1}{x}\right) - \frac{2y}{x^3}(1 - \ln x) = \frac{y}{x^3}(2 \ln x - 3); \qquad f_{yy} = 0.$

$f_{xy} = \frac{1}{x^2}(1 - \ln x); \qquad f_{yx} = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \ln x = \frac{1}{x^2}(1 - \ln x).$

So $f_{xy} = f_{yx}$.

4. $f(x,y) = \frac{1}{x^2 + y^2}.$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.$

$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+y^2)^2} = \frac{-2r \cos t}{r^4}; \ \frac{\partial x}{\partial t} = -r \sin t;$

$\frac{\partial f}{\partial y} = \frac{-2y}{(x^2+y^2)^2} = \frac{-2r \sin t}{r^4}; \ \frac{\partial y}{\partial t} = r \cos t.$

Hence, $\frac{\partial f}{\partial t} = \frac{2r^2 \sin t \cos t}{r^4} - \frac{2r^2 \sin t \cos t}{r^4} = 0.$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}.$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are as above. $\frac{\partial x}{\partial r} = \cos t$, $\frac{\partial y}{\partial r} = \sin t$

Hence, $\frac{\partial f}{\partial r} = \frac{-2r \cos^2 t - 2r \sin^2 t}{r^4} = \frac{-2(\cos^2 t + \sin^2 t)}{r^3} = \frac{-2}{r^3}.$

5. $f(x,y) = x^2 + xy - y^2$.

(i) $f(r,\theta) = (r\cos\theta)^2 + (r\cos\theta)\,(r\sin\theta) - (r\sin\theta)^2$

$= r^2(\cos^2\theta + \cos\theta\sin\theta - \sin^2\theta)$.

$\dfrac{\partial f}{\partial r} = 2r\,(\cos^2\theta + \cos\theta\sin\theta - \sin^2\theta)$.

$\dfrac{\partial f}{\partial \theta} = r^2\,(-2\cos\theta\sin\theta + \cos\theta\cos\theta - \sin\theta\sin\theta - 2\sin\theta\cos\theta)$

$= r^2\,(\cos^2\theta - \sin^2\theta - 4\cos\theta\sin\theta)$.

(ii) $\dfrac{\partial x}{\partial r} = \cos\theta;\quad \dfrac{\partial x}{\partial \theta} = -r\sin\theta;\quad \dfrac{\partial y}{\partial r} = \sin\theta;\quad \dfrac{\partial y}{\partial \theta} = r\cos\theta.$

By the chain rule $\dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} = (2x+y)\,\cos\theta + (x-2y)\,\sin\theta$

$= (2r\cos\theta + r\sin\theta)\,\cos\theta + (r\cos\theta - 2r\sin\theta)\,\sin\theta$

$= 2r(\cos^2\theta + \cos\theta\sin\theta - \sin^2\theta)$.

$\dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \theta} = (2x+y)\,(-r\sin\theta) + (x-2y)\,r\cos\theta$

$= (2r\cos\theta + r\sin\theta)\,(-r\sin\theta) + (r\cos\theta - 2r\sin\theta)\,r\cos\theta$

$= r^2\,(\cos^2\theta - \sin^2\theta - 4\cos\theta\sin\theta)$.

6. $f(x,y) = x^3y - y^3x;\quad x = uv;\quad y = \dfrac{u}{v}.$

$\dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u};\quad \dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}.$

$\dfrac{\partial f}{\partial x} = 3x^2y - y^3 = 3u^2v^2\dfrac{u}{v} - \dfrac{u^3}{v^3} = 3u^3v - \dfrac{u^3}{v^3}.$

$\dfrac{\partial f}{\partial y} = x^3 - 3y^2x = \dfrac{u^3}{v^3} - 3\dfrac{u^2}{v^2}uv = u^3v^3 - \dfrac{3u^3}{v}.$

$\dfrac{\partial x}{\partial u} = v;\quad \dfrac{\partial x}{\partial v} = u;\quad \dfrac{\partial y}{\partial u} = \dfrac{1}{v};\quad \dfrac{\partial y}{\partial v} = \dfrac{-u}{v^2}.$

$\dfrac{\partial f}{\partial u} = \left(3u^3v - \dfrac{u^3}{v^3}\right)v + \left(u^3v^3 - \dfrac{3u^3}{v}\right)\dfrac{1}{v}$

$= 3u^3v^2 - \dfrac{u^3}{v^2} + u^3v^2 - \dfrac{3u^3}{v^2} = 4u^3v^2 - \dfrac{4u^3}{v^2}.$

$$\frac{\partial f}{\partial v} = \left(3u^3v - \frac{u^3}{v^3}\right)u + \left(u^3v^3 - \frac{3u^3}{v}\right)\frac{-u}{v^2}$$
$$= 3u^4v - \frac{u^4}{v^3} - u^4v + \frac{3u^4}{v^3} = 2u^4v + \frac{2u^4}{v^3}.$$

7. $f(x, y, z) = 2y - \sin(xz),\ x = 3t,\ y = \mathrm{e}^{t-1},\ z = \ln t.$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$
$$\frac{\partial f}{\partial x} = -z\cos(xz); \quad \frac{\partial f}{\partial y} = 2; \quad \frac{\partial f}{\partial z} = -x\cos(xz).$$
$$\frac{dx}{dt} = 3; \quad \frac{dy}{dt} = \mathrm{e}^{t-1}; \quad \frac{dz}{dt} = \frac{1}{t}.$$
$$\frac{\partial f}{\partial t} = -3z\cos(xz) + 2\mathrm{e}^{t-1} - \frac{x\cos(xz)}{t}$$
$$= -3\ln t\cos(3t\ln t) + 2\mathrm{e}^{t-1} - \frac{3t\cos(3t\ln t)}{t}$$
$$= -3\cos(3t\ln t)(1 + \ln t) + 2\mathrm{e}^{t-1}.$$

8. $f(x, y) = x^2 + xy + y^2,\ x = uv,\ y = u/v.$

To show that $uf_u + vf_v = 2xf_x$ and $uf_u - vf_v = 2yf_y$
we need to find f_u, f_v, f_x and f_y.

$$f_u = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}; \quad f_v = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$
$$f_u = (2x + y)(v) + (x + 2y)(\frac{1}{v}) = 2uv^2 + 2u + \frac{2u}{v^2}.$$
$$f_v = (2x + y)(u) + (x + 2y)(\frac{-u}{v^2}) = 2u^2v - \frac{2u^2}{v^3}.$$
$$f_x = 2x + y = 2uv + \frac{u}{v}. \text{ So, } 2xf_x = 2uvf_x = 4u^2v^2 + 2u^2.$$
$$f_y = x + 2y = uv + \frac{2u}{v}. \text{ So, } 2yf_y = \frac{2u}{v}f_x = 2u^2 + \frac{4u^2}{v^2}.$$
Now $uf_u + vf_v = 2u^2v^2 + 2u^2 + \frac{2u^2}{v^2} + 2u^2v^2 - \frac{2u^2}{v^2}$
$= 4u^2v^2 + 2u^2 = 2xf_x$ as required,

and $uf_u - vf_v = 2u^2v^2 + 2u^2 + \dfrac{2u^2}{v^2} - 2u^2v^2 + \dfrac{2u^2}{v^2}$

$= 2u^2 + \dfrac{4u^2}{v^2} = 2yf_y$ as required.

9. $u(x,y) = \ln(1+xy^2)$.

$$\frac{\partial u}{\partial x} = \frac{1}{1+xy^2}\,.\,y^2 = \frac{y^2}{1+xy^2}; \qquad \frac{\partial^2 u}{\partial x^2} = \frac{-y^2\,.\,y^2}{(1+xy^2)^2} = -\frac{y^4}{(1+xy^2)^2}.$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(1+xy^2)\,.\,2y - 2xy\,.\,y^2}{(1+xy^2)^2} = \frac{2y}{(1+xy^2)^2}.$$

Hence $2\dfrac{\partial^2 u}{\partial x^2} + y^3\dfrac{\partial^2 u}{\partial y \partial x} = -\dfrac{2y^4}{(1+xy^2)^2} + \dfrac{2y^4}{(1+xy^2)^2} = 0.$

10. $u(x,y) = x^2\cosh(xy^2+1)$.

NOTE. $\dfrac{d}{dx}(\sinh x) = \cosh x; \dfrac{d}{dx}(\cosh x) = \sinh x.$

$$\frac{\partial u}{\partial x} = x^2\sinh(xy^2+1)\,.\,y^2 + 2x\cosh(xy^2+1)$$

$$= x^2y^2\sinh(xy^2+1) + 2x\cosh(xy^2+1)$$

$$\frac{\partial u}{\partial y} = x^2\sinh(xy^2+1)\,.\,2xy = 2x^3y\sinh(xy^2+1)$$

Hence

$$2x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 2x^3y^2\sinh(xy^2+1) + 4x^2\cosh(xy^2+1) - 2x^3y^2\sinh(xy^2+1)$$

$$= 4x^2\cosh(xy^2+1) = 4u.$$

11.

$$\frac{\partial w}{\partial t} = \frac{1}{2x+2ct}2c \qquad \frac{\partial^2 w}{\partial t^2} = \frac{-4c^2}{(2x+2ct)^2}$$

$$\frac{\partial w}{\partial x} = \frac{1}{2x+2ct}2 \qquad \frac{\partial^2 w}{\partial x^2} = \frac{-4}{(2x+2ct)^2}$$

Hence $\dfrac{\partial^2 w}{\partial t^2} = c^2\dfrac{\partial^2 w}{\partial x^2}.$