On the Conjectured Upper Bounds for Entries of Mutation Count Matrices

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ABSTRACT. Simple expressions for the previously given conjectural upper bounds for the entries of mutation count matrices are presented.

1. Introduction

For an introduction to the topic of oriented matroids, see [2].

Let $\mathcal{O}$ denote an oriented matroid of rank $r = d+1$ without loops. It is associated with an arrangement of finitely many pseudospheres in the d-sphere $\mathrm{S}^d \subseteq R^r$. Let n be the number of elements of the underlying matroid, so that n is the number of pseudospheres in the arrangement, and let these be denoted by H_i ($1 \le i \le n$). Let H_i^+ and H_i^- denote the two open *sides* of H_i, each assumed to be an open ball; for each i the sets H_i, H_i^+, and H_i^- form a partition of S^d. The minimal nonempty sets of the form $\bigcap_{i=1}^n K_i$, where for each i, K_i is either H_i, H_i^+, or H_i^-, are the *cells* of the arrangement. A pseudosphere H_i *supports* the cell C if the pseudosphere and the topological closure of the cell have nonempty intersection.

We describe the notion of a "mutation" of a uniform oriented matroid, and then, that of the "mutation count matrix" of an ordered pair of uniform oriented matroids having common rank and underlying set.

Any d-dimensional cell of the arrangement is contained in exactly one of the sides, H_i^+ or H_i^-, of each pseudosphere H_i. Therefore each d-cell C determines an element $v = v(C) \in \{1, -1\}^n$, where v_i is 1 if $C \subseteq H_i^+$, and -1 if $C \subseteq H_i^-$. Such a vector is called a *tope* of the oriented matroid. The oriented matroid is uniquely determined by its set of topes.

Let $\mathcal{T}$ denote the set of topes of a uniform oriented matroid, $\mathcal{O}$. If v is a tope, then $-v$ is a tope. It is sometimes possible to replace topes v and $-v$ by two other elements v' and $-v'$ of $\{1, -1\}^n$, preserving the property that the new set $\mathcal{T}'$ is the set of topes of a uniform oriented matroid. This is the case precisely when the d-cell C for which $v = v(C)$ has exactly $d+1$ supporting pseudospheres H_i. In this case, the d-cell is a *simplex cell* of the arrangement and the tope is termed *simplicial*. Of course, $-v$ will also be a simplicial tope, and v' and $-v'$ will be simplicial topes of the new oriented matroid. The new oriented matroid, $\mathcal{O}'$, is said to be obtained from $\mathcal{O}$ by a *mutation*. The *reverse* of this mutation is the mutation that transforms $\mathcal{O}'$ into $\mathcal{O}$.

It is not difficult to describe the new topes v' and $-v'$. Let $K \subseteq [n]$ denote the set of indices i such that the pseudosphere H_i supports the simplicial tope v. Since v is simplicial, there are exactly $r = d+1$ of these. One of the two new topes is v', where v'_i equals $-v_i$ if $i \in K$, and equals v_i, otherwise; the other is its negative, $-v'$.

Let $L \subseteq [n]$ denote the set of the remaining $n-r$ elements of $[n]$, $L = [n] \setminus K$. Let k denote the number of indices $i \in K$ such that $v_i = -1$, and let ℓ denote the number of indices $i \in L$ such that $v_i = -1$. We term the pair (k, ℓ) the *type* of the simplicial tope v. Then $-v$ has type $(r-k, n-r-\ell)$. Also, in $\mathcal{O}'$, v' and $-v'$ have types $(r-k, \ell)$ and $(k, n-r-\ell)$. The *type* of the mutation taking $\mathcal{O}$ to $\mathcal{O}'$ is designated by either of the pairs (k, ℓ), $(r-k, n-r-\ell)$. Its reverse has the types $(r-k, \ell)$ and $(k, n-r-\ell)$.

Given a sequence of uniform oriented matroids $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_m = \mathcal{O}''$ which transforms the oriented matroid $\mathcal{O}$ into $\mathcal{O}''$, the entry $M_{k,\ell}$, where $0 \leq k \leq \lfloor \frac{r-1}{2} \rfloor$ and $0 \leq \ell \leq \lfloor \frac{n-r-1}{2} \rfloor$, of the *mutation count matrix* $M = M(\mathcal{O}, \mathcal{O}'')$ records the number $c_+ - c_-$, where c_+ is the number of mutations of type (k, ℓ) in the sequence, while c_- is the number of reverse mutations of type (k, ℓ). It is shown in [6] that this number only depends upon the pair $\mathcal{O}$ and $\mathcal{O}''$, not upon the particular sequence. However, it is not known whether or not, given two uniform oriented matroids of the same rank on the same underlying set, there must exist such a sequence of mutations connecting them (although this is always so in the realizable case), so the definition given in [6], which will be repeated here, differs from the foregoing description. (Perhaps the mutation count matrices should be called "mutation pseudo-count matrices"!)

For each i, let x_i and y_i be indeterminates. For each cell C let w_C denote the product

$$w_C = \Big(\prod_{i \in \Lambda^+} x_i\Big)\Big(\prod_{j \in \Lambda^-} y_j\Big)$$

where $\Lambda^+ = \{i : C \subseteq H_i^+\}$ and $\Lambda^- = \{j : C \subseteq H_i^-\}$. The *total polynomial* (see [5]) of $\mathcal{O}$ is

$$T_{\mathcal{O}}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \sum_{C,\text{ a cell}} w_C.$$

The *(dual) Radon catalog* is

$$R_{\mathcal{O}}(x, y) = T_{\mathcal{O}}(x, y, x, y, \ldots, x, y).$$

It is a polynomial of degree n. When $\mathcal{O}$ is a uniform oriented matroid (which is assumed to be the case henceforth), it has no terms of total degree less than $n-r+1$. The coefficient of $x^k y^\ell$ is the number of cells of the arrangement which are on the positive side of k pseudospheres and on the negative side of ℓ pseudospheres.

For $0 \leq k \leq \lfloor \frac{r-1}{2} \rfloor$ and $0 \leq \ell \leq \lfloor \frac{n-r-1}{2} \rfloor$, let $m_{k,\ell}$ denote the *little mutation polynomial*

$$m_{k,\ell}(x, y) = (x^k y^{r-k} - x^{r-k} y^k)((1+x)^\ell (1+y)^{n-r-\ell} - (1+x)^{n-r-\ell}(1+y)^\ell).$$

It is shown in [6] that, for any pair $\mathcal{O}$, $\mathcal{O}''$ of uniform oriented matroids of rank r on the same underlying set of cardinality n, there are unique integers $\gamma_{k,\ell}$ such that

$$R_{\mathcal{O}''}(x, y) - R_{\mathcal{O}}(x, y) = \sum \gamma_{k,\ell} m_{k,\ell}. \tag{1}$$

The matrix $(\gamma_{k,\ell})$ is the *mutation count matrix* $M(\mathcal{O}, \mathcal{O}'')$. If there exists a sequence of mutations which transforms $\mathcal{O}$ into $\mathcal{O}''$ (as will be the case if the well–known conjecture of Cordovil and Las Vergnas holds), then the (k, ℓ)–th entry of $M(\mathcal{O}, \mathcal{O}'')$ enumerates mutations of type (k, ℓ) in the sequence, in the way described above.

In [6] it is conjectured that the entries in $M(\mathcal{O}, \mathcal{O}'')$ are bounded above by the numbers $\delta_{k,\ell}$:

$$\delta_{k,\ell} = \sum_{i=0}^{k} \sum_{j=0}^{\ell} (-1)^{k-i+\ell-j} \binom{r-i}{k-i} \binom{n-r-j}{\ell-j} \binom{n}{i, j, n-i-j}, \tag{2}$$

for $0 \leq k \leq \lfloor \frac{r-1}{2} \rfloor$ and $0 \leq \ell \leq \lfloor \frac{n-r-1}{2} \rfloor$. These numbers are obtained as the entries in the mutation count matrix $M(\widehat{\mathcal{A}}(n, n-r), \mathcal{A}(n, r))$ where $\mathcal{A}(n, r)$ is the alternating oriented matroid of rank r and $\widehat{\mathcal{A}}(n, n-r)$ is the dual of the alternating oriented matroid of rank $n-r$.

The expression above leaves a little to be desired: It is not even clear from (2) that the $\delta_{k,\ell}$'s are nonnegative, which must be the case if the conjecture is valid, since $M(\mathcal{O}, \mathcal{O}'') = -M(\mathcal{O}'', \mathcal{O})$. As it happens, there are closed–form expressions for the numbers $\delta_{k,\ell}$.

In this paper we give simple expressions for the $\delta_{k,\ell}$'s, namely, letting $\hat{r}$ denote $n-r$ (which is the rank of the dual oriented matroid) and $\hat{d}=\hat{r}-1=n-r-1$,

$$\delta_{k,\ell}=\binom{\hat{d}-k+\ell}{\ell}\binom{d-\ell+k}{k}-\binom{\hat{d}-k+\ell}{\ell-1}\binom{d-\ell+k}{k-1} \tag{3}$$

and

$$\delta_{k,\ell}=\frac{\hat{r}r-\ell\hat{r}-kr}{(\hat{r}-k+\ell)(r-\ell+k)}\binom{\hat{r}-k+\ell}{\ell}\binom{r-\ell+k}{k}. \tag{4}$$

It is easily seen (and left to the reader) that these two expressions yield the same numbers. From the second of these, it is clear that the $\delta_{k,\ell}$'s are nonnegative, when k and ℓ are in the given range. The equations hold more generally, however, for any nonnegative integers k and ℓ. For this extension we use the convention that $\binom{x}{0}=1$, $\binom{x}{m}$ represents the polynomial function of x given as

$$\binom{x}{m}=\frac{x}{1}\frac{x-1}{2}\cdots\frac{x-m+1}{m}$$

for m a positive integer, and $\binom{x}{m}=0$ if m is a negative integer. Then the equations (2), (3), and (4) are equivalent.

2. Generating Functions and Proofs

Let $f(x,y)$ be the rational function

$$f(x,y)=\frac{(1+x+y)^n}{(1+x)^{r-k+1}(1+y)^{n-r-\ell+1}}. \tag{5}$$

It is not difficult to see that $\delta_{k,\ell}$ as given in (2) above is the coefficient of $x^k y^\ell$ in the the power series expansion of $f(x,y)$:

$$f(x,y)=\sum_{k,\ell\geq 0}\delta_{k,\ell}x^k y^\ell. \tag{6}$$

It would be nice to have a single function from which to extract all the coefficients $\delta_{k,\ell}$. Such a function, given in Proposition 3, will be obtained through the use of the following two lemmas.

Lemma 1. *Let $h(x)$ be a formal power series in x, with coefficients in a commutative ring. Then the coefficients of x^i in*

$$\frac{h(x)}{(1+x)^{a-i+1}} \qquad \textit{and} \qquad (1-x)^a h\left(\frac{x}{1-x}\right)$$

are equal.

Proof. This is clear when $h(x) = x^m$ for some nonnegative integer m, the common value of the coefficients then being $(-1)^{i-m}\binom{a-m}{i-m}$. The general result follows by additivity. $\square$

With two variables, this appears as follows.

Lemma 2. *Let $h(x, y)$ be a formal power series in x, y with coefficients in a commutative ring. Then the coefficients of $x^k y^\ell$ in*

$$\frac{h(x, y)}{(1+x)^{a-k+1}(1+y)^{b-\ell+1}}$$

and

$$(1-x)^a(1-y)^b h\left(\frac{x}{1-x}, \frac{y}{1-y}\right)$$

are equal.

Proof. Use Lemma 1 twice. $\square$

Proposition 3. *The coefficients of $x^k y^\ell$ in the power series expansions for $\frac{(1+x+y)^n}{(1+x)^{r-k+1}(1+y)^{n-r-\ell+1}}$ and $\frac{(1-xy)^n}{(1-x)^{n-r}(1-y)^r}$ are equal.*

Proof. Use Lemma 2 with the function $h(x, y) = (1 + x + y)^n$, $a = r$, and $b = n - r$. $\square$

It follows that $\delta_{k,\ell}$ is the coefficient of $x^k y^\ell$ in the power series expansion for $\frac{(1-xy)^n}{(1-x)^{n-r}(1-y)^r}$.

Let $G_{a,b}$ be the function

$$G_{a,b}(x, y) = \frac{(1-xy)^{a+b-1}}{(1-x)^a(1-y)^b}. \tag{7}$$

We wish to show that the coefficient of $x^k y^\ell$ in the power series expansion of $G_{a,b}$ about the origin is given by the product $\binom{a-1-\ell+k}{k}\binom{b-1-k+\ell}{\ell}$. The following lemma gives this for $a = 0$.

Lemma 4. *We have the power series expansion*

$$\frac{(1-xy)^{b-1}}{(1-y)^b} = \sum_{k,\ell \geq 0} \binom{k-\ell-1}{k}\binom{b-k+\ell-1}{\ell} x^k y^\ell.$$

Proof. We begin with

$$(1-xy)^{b-1} = \sum_{k=0}^{b-1} (-1)^k \binom{b-1}{k} x^k y^k$$

and

$$\frac{1}{(1-y)^b} = \sum_{m=0}^{\infty} \binom{b-1+m}{b-1} y^m.$$

Upon multiplication we get

$$\sum_{k,\ell\geq 0}(-1)^k\binom{b-1}{k}\binom{b-1+\ell-k}{b-1}x^ky^\ell.$$

We have

$$\binom{b-1}{k}\binom{b-1+\ell-k}{b-1}=\frac{(b-1)!}{k!(b-k-1)!}\,\frac{(b-1+\ell-k)!}{(b-1)!(\ell-k)!}.$$

Upon replacing the $(b-1)!$'s by $\ell!$'s and rearranging, we get

$$\frac{\ell!}{k!(\ell-k)!}\,\frac{(b-1+\ell-k)!}{\ell!(b-k-1)!}=\binom{\ell}{k}\binom{b-1+\ell-k}{\ell}.$$

Finally, rewriting $\binom{\ell}{k}$ as $(-1)^k\binom{k-\ell+1}{k}$ yields the desired conclusion. □

Lemma 5. *When*

$$\alpha_{k,\ell}=\binom{a-\ell+k}{k}\binom{b-1-k+\ell}{\ell}$$

and

$$\beta_{k,\ell}=\binom{a-1-\ell+k}{k}\binom{b-k+\ell}{\ell}$$

we have

$$\alpha_{k,\ell}-\alpha_{k-1,\ell}=\beta_{k,\ell}-\beta_{k,\ell-1}.$$

Proof. Both sides are equal to

$$\binom{a-\ell+k-1}{a-\ell}\binom{b-1-k+\ell}{\ell}\frac{ab-ak-b\ell}{k(b-k)}.$$

□

Theorem 6. *If $G_{a,b}$ is as in (7) then*

$$G_{a,b}(x,y)=\sum_{k,\ell\geq 0}\binom{a-1-\ell+k}{k}\binom{b-1-k+\ell}{\ell}x^ky^\ell.$$

Proof. The proof proceeds by induction on a, the case $a=0$ having been dealt with in Lemma 4. Suppose $a\geq 0$ and that the result holds for a.

By the inductive assumption, the coefficient of x^ky^ℓ in $G_{a,b+1}(x,y)$ is the $\beta_{k,\ell}$ of Lemma 5. Let $\tilde\alpha_{k,\ell}$ be that of $G_{a+1,b}$. We must show that $\tilde\alpha_{k,\ell}=\alpha_{k,\ell}$. This is certainly the case for $k=0$, for $G_{a+1,b}(0,y)=\frac{1}{(1-y)^b}$, so that the coefficient of y^ℓ is $\tilde\alpha_{0,\ell}=\binom{\ell+b-1}{\ell}=\alpha_{0,\ell}$. Then $\tilde\alpha_{k,\ell}$ and $\alpha_{k,\ell}$ agree for $k=0$.

Also we have $(1-x)G_{a+1,b}(x,y)=(1-y)G_{a,b+1}(x,y)$, so $\tilde\alpha_{k,\ell}-\tilde\alpha_{k-1,\ell}=\beta_{k,\ell}-\beta_{k,\ell-1}$. By Lemma 5, $\tilde\alpha_{k,\ell}-\tilde\alpha_{k-1,\ell}=\alpha_{k,\ell}-\alpha_{k-1,\ell}$ for all

$k \geq 1, \ell \geq 0$. Since $\tilde{\alpha}_{0,\ell} = \alpha_{0,\ell}$ for $\ell \geq 0$, it follows that $\tilde{\alpha}_{k,l} = \alpha_{k,l}$ for all $k, \ell \geq 0$, completing the inductive step. (We note the resemblance to a Wilf–Zeilberger proof; see [8].)

This theorem remains valid when a and b are not nonnegative integers, the power series converging inside the unit polydisk. To see this, note that for fixed k and ℓ, the coefficients in the Taylor series will be polynomials in a and b, and, since these polynomials agree for nonnegative integers a and b, they agree everywhere. □

Theorem 7. *The entries $\delta_{k,\ell}$ are as given in (3).*

Proof. Use Theorem 6 with $a = r$ and $b = n - r$, noting that $\delta_{k,\ell}$ is the coefficient of $x^k y^\ell$ in $(1 - xy)G_{r,n-r}(x, y)$. □

3. Various Questions and Special Cases

In the case of mutation count matrices of realizable oriented matroids, the conjectured inequalities hold for the first row and column as a consequence of the "g–theorem" characterizing the f–vectors of simplicial polytopes, proven by Stanley. (See [9].) In general, the conjecture is open, even in the realizable case.

For $r = 3$, the conjecture would imply that of Guy that the spherical crossing number of K_n is equal to $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$. (The same number is conjectured for the value of the topological crossing number of K_n. In this case, no refinement of the sort described here is known.)

It would certainly be nice to be able to state conditions characterizing the mutation count matrices. The difficulty of this problem is indicated by recent results on some old problems concerning configurations of planar point sets. Let S denote a set of $2n$ points in the plane, no three colinear. Enumerate the lines L which are determined by pairs of the points and for which the same number of points of S lie in each of the open halfplanes determined by L. How large can this number be? This problem dates to [4] and [7]. For some recent lower and upper bounds, see [10] and [3]. Certainly a nice characterization of the mutation count matrices of acyclic, rank 3 oriented matroids would bear on this problem. The fact that there is still a large gap between the known upper and lower bounds seems to show that even making reasonable guesses will be hard.

The problem of determining the rectilinear crossing number of K_n also illustrates this. Although there has been a flurry of activity and considerable progress on this problem, it is difficult to find a reasonable guess about what might be the answer in general. For a web site devoted to this problem, see [1].

It would be nice to have a direct combinatorial proof that the numbers given in (3) are the entries of the mutation count matrix for the alternating and dual–alternating uniform oriented matroids. In this case, there certainly exist sequences of mutations connecting the two oriented matroids; and it might be possible to find a particular such sequence, and directly enumerate the mutations of each type. This would yield a different proof of Theorem 7.

References

[1] O. Aichholzer, web page: http://www.ist.tugraz.at/staff/aichholzer/crossings.html.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics, Cambridge University Press, Cambridge, 1992.

[3] T. K. Dey, Improved bounds for planar k-sets and related problems. *Discrete Comput. Geom.* **19** (1998), no. 3, Special Issue, 373–382.

[4] P. Erdös, L. Lovász, A. Simmons, E. G. Straus, Dissection graphs of planar point sets. In *A Survey of Combinatorial Theory* (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), pp. 139–149. North-Holland, Amsterdam, 1973.

[5] J. Lawrence, Total polynomials of uniform oriented matroids. *European Journal of Combinatorics* **21** (2000), 3–12.

[6] J. Lawrence, Mutation polynomials and oriented matroids. *Discrete Comput. Geom.* **24** (2000), 365–389.

[7] L. Lovász, On the number of halving lines. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **14** (1971), 107–108.

[8] M. Petkovsek, H. Wilf, D. Zeilberger, $A = B$. A. K. Peters, Ltd., Wellesley, MA, 1996.

[9] R. P. Stanley, *Combinatorics and Commutative Algebra*, second edition. Birkhauser Boston, Inc., Boston, MA, 1996.

[10] G. Toth, Point sets with many k-sets. *Discrete Comput. Geom.* **26** (2001), no. 2, 187–194.