NAME (print): Key

Math 114 Fall 2002—Exam 3

Work carefully and neatly and remember that I cannot grade what I cannot read. You must show all relevant work in the appropriate space. You may receive no credit for a correct answer if there is insufficient supporting work. Notes and books are NOT allowed.

[40] 1. Determine if the following series converge or diverge (you must justify your answer).

(a) $\sum_{n=0}^{\infty} \frac{7 \cdot 4^n}{5^n}$ - Geometric Series with $r = \frac{4}{5}$ & $a = 7$

(b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

$0 \leq \sin^2 n \leq 1$

So $0 \leq \frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ ← P series with $p>1$. so it converges

Hence the original series converges

(c) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$\ln n > 1$ for $n > e$

So $\frac{\ln n}{n} > \frac{1}{n}$ ← Diverges

Hence original series diverges

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{\cancel{1 \cdot 3 \cdot 5} \cdots \cancel{(2n-1)}(2n+1)} \cdot \frac{\cancel{1 \cdot 3 \cdot 5} \cdots \cancel{(2n-1)}}{\cancel{n!}}$

$= \frac{n+1}{2n+1} \longrightarrow \frac{1}{2} < 1$

So series converges

[12] 2. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ converges. Then determine how many terms of the series you would need to add in order to have the error less than 0.1.

A.S.T. $\frac{n}{n^2+1}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{\frac{1}{n}}{1+\frac{1}{n^2}} \longrightarrow \frac{0}{1}=0$

$f(x)=\frac{x}{x^2+1}$, $f' = \frac{(x^2+1)\cdot 1 - 2x(x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$

So a_n is decreasing

Hence by AST, series converges

$a_{10} = \frac{10}{101} < .1$ So we need to add the first $\underline{9 \text{ term}}$

[8] 3. Let $f(x)$ be a function such that $f'(x) = 3f(x)$. If $f(0) = 2$, find the Maclaurin series for this function. (Give a formula for the nth term.)

$f(0)=2$

$f'(x)=3f(x)$

$f''(x)=3^2f(x)$

$f^{(n)}(x)=3^nf(x)$

$c_n = \frac{f^{(n)}(0)}{n!} = \frac{3^n\cdot 2}{n!}$

$$\boxed{\sum_{n=0}^{\infty}\frac{2\cdot 3^n}{n!}x^n}$$

[16] 4. Find the Maclaurin series for the following functions (you should use a series which we have already derived).

(a) $\frac{1}{1+x^3}$

$\frac{1}{1-x} = 1+x+x^2+x^3$

$\frac{1}{1+x^3} = 1-x^3+x^6-x^9\ldots$

$$\boxed{=\sum_{n=0}^{\infty}(-1)^nx^{3n}}$$

(b) $\int\frac{\sin x}{x}$

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots$

$\frac{\sin x}{x} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}\ldots$ | So $\int\frac{\sin x}{x} = x-\frac{x^3}{3\,3!}+\frac{x^5}{5\cdot 5!}-\ldots$

$=$

[12] 5. Find the radius and then the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+2)^{\cancel{n+1}}}{3^{\cancel{n+1}}\cdot(n+1)} \cdot \frac{\cancel{3^n}\, n}{(x+2)^{\cancel{n}}}\right|$$

$$= \left|\frac{n}{n+1}\right| \left|\frac{x+2}{3}\right| \rightarrow 1 \quad \frac{1}{3}|x+2| < 1 \text{ or } |x+2| < 3$$

$\boxed{\text{So } R = 3}$ $\quad -3 < x+2 < 3$ or $-5 < x < 1$

$(-5,1)$ - check end points

$x = 1$ - $\sum \frac{3^n}{3^n \cdot n} = \sum \frac{1}{n}$ - Diverges

$x = -5$ - $\sum \frac{(-3)^n}{3^n \cdot n} = \sum \frac{(-1)^n}{n}$ - converges

$\boxed{[-5, 1)}$

[12] 6. Write the first four terms of the Taylor series for $f(x) = \sqrt{x}$ about $a = 4$ (Three points extra credit if if you can write a the general formula).

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

at $x = 4$

$f(4) = 2$

$f'(4) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$f''(4) = -\frac{1}{4} \cdot \frac{1}{\sqrt{4^3}}$

$= -\frac{1}{4} \cdot \frac{1}{2^3}$

$f'''(4) = \frac{3}{8} \cdot \frac{1}{\sqrt{4^5}}$

$= \frac{3}{2^3} \cdot \frac{1}{2^5}$

$\cancel{2 + \frac{1}{2^2} - \frac{1}{2^5 \cdot 2!} + \frac{3}{2^8 3!}}$

$$\boxed{2 + \frac{1}{2^2}(x-4) - \frac{1}{2^5 2!}(x-4)^2 + \frac{3}{2^8 3!}(x-4)^3}$$

$\cancel{2 + \sum_{n=} \frac{1\cdot 3\cdot 5 \,(x-4)^n}{2^{3n-1}\, n!}}$

$$2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5 \cdots (2n-3)\,(x-4)^n}{2^{3n-1}\cdot n!}$$

Extra Credit