NAME (print): Key

Math 114 Fall 2002—Exam 2

Instructor: J Shapiro

Work carefully and neatly and remember that I cannot grade what I cannot read. You must show all relevant work in the appropriate space. You may receive no credit for a correct answer if there is insufficient supporting work. Notes, books and graph are NOT allowed.

[8] 1. ~~Set up, but do not compute the integral to compute the length of the curve $y = \ln(\sin x)$ $\pi/6 \le x \le \pi/3$.~~

$$\int_0^3 \frac{1}{x\sqrt{x}}\,dx = \int_0^3 x^{-3/2}\,dx$$

$$= -2x^{-1/2}\Big|_0^3 = \frac{-2}{\sqrt{x}}\Big|_0^3 = \frac{-2}{0} + \frac{2}{\sqrt{3}} = \infty \quad \boxed{\text{- Diverges}}$$

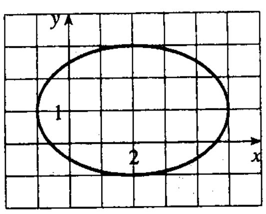
[8] 2. Set up, but do not do the final sum to compute $\int_{-1}^{1} \sqrt{1+x^3}dx$ using the Trapezoidal Rule with $n = 4$.

$$\Delta x = \frac{1-(-1)}{4} = \frac{1}{2}$$

So $$\int_{-1}^{1} \sqrt{1+x^3}\,dx \approx \frac{\frac{1}{2}}{2}\left[f(-1) + 2f(-.5) + 2f(0) + 2f(.5) + f(1)\right]$$

$$\boxed{= \frac{1}{4}\left[0 + 2\sqrt{\tfrac{7}{8}} + 2 + 2\sqrt{\tfrac{9}{8}} + \sqrt{2}\,\right]}$$

[8] 3. Find the the equation of the ellipse drawn below and then find its foci.



General form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$$\boxed{\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1}$$

foci: $c^2 = a^2 - b^2 = 9 - 4 = 5$

$c = \pm\sqrt{5}$

foci
$(2+\sqrt{5}, 1)$
$(2-\sqrt{5}, 1)$

[8] 4. Determine whether the following integral is convergent or divergent. If it is convergent, give its value.

$$\int_1^{\infty} \frac{1}{(2x+1)^3}dx$$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int_1^{\infty} \frac{1}{u^3}\,du$$

$$= \frac{1}{2}\, u^{-2}\left(-\frac{1}{2}\right)$$

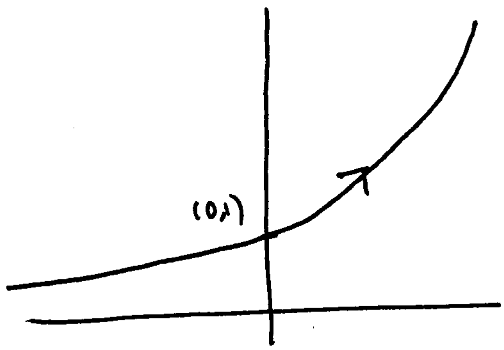
$$= -\frac{1}{4}\left[\frac{1}{(2x+1)^2}\right]_1^{\infty}$$

$$= -\frac{1}{4}\left[0 - \frac{1}{9}\right] = \boxed{\frac{1}{36} \text{ - Converges}}$$

[22] 5. Consider the curve which is defined parametrically by $x = \ln t, y = \sqrt{t}$.

(a) Sketch the curve for $t > 0$ and indicate with an arrow the direction which the curve is traced as the parameter increases.

t	x	y
$1/2$	$\ln 1/2$	$\sqrt{1/2}$
1	0	1
2	$\ln 2$	$\sqrt{2}$
4	$\ln 4$	2



when $0 < t < 1$

$\ln t < 0$

(b) Find the equation of the tangent line to this curve when $t = 1$ (in Cartesian coordinates).

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}t^{-1/2}}{\frac{1}{t}} = \frac{1}{2}\,\frac{\frac{1}{t^{1/2}}}{1/t} = \frac{1}{2}t^{1/2}$$

when $t = 1$, $m = \frac{1}{2}$ $\quad x = 0, y = 1$

$$\boxed{y - 1 = \tfrac{1}{2}(x - 0)}$$

(c) Eliminate the parameter to find a Cartesian equation of the curve.

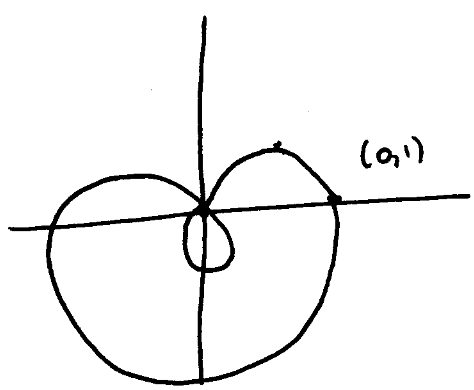
$x = \ln t \qquad y = \sqrt{t}$

$e^x = t$

so $\boxed{y = \sqrt{e^x} = e^{\frac{1}{2}x}}$

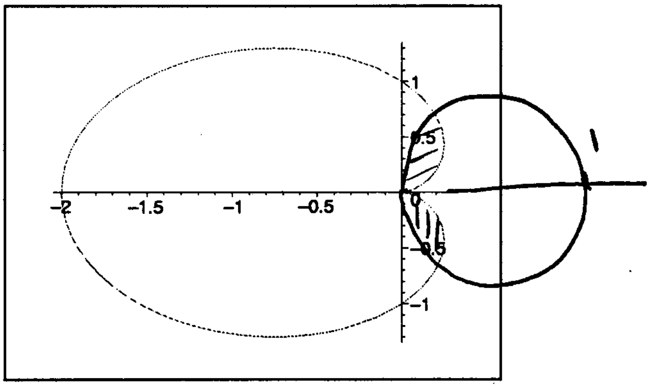
[8] 6. Sketch the polar curve $r = 1 - 2\sin\theta$ for $0 \le \theta \le 2\pi$.

θ	r
0	1
$\pi/6$	0
$\pi/2$	-1



[14] 7. Below is the graph of $r = 1 - \cos\theta$.

(a) On the same axis draw the graph of $r = \cos\theta$.



(b) Setup the integral to compute the area inside of both curves (don't evaluate).

$1 - \cos\theta = \cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3$

$\theta = -\pi/3$

Top half:

$\frac{1}{2}\int_0^{\pi/3} (1-\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (\cos^2\theta) d\theta$

Total = 2 times top

$\boxed{= \int_0^{\pi/3} (1-\cos\theta)^2 \, d\theta + \int_{\pi/3}^{\pi/2} \cos^2\theta \, d\theta}$

8. Determine if the sequence $\left\{ \frac{\sqrt{n}}{2+\sqrt{n}} \right\}$ converges and if it converges, find the limit.

$\frac{\sqrt{n}}{2+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \frac{1}{\frac{2}{\sqrt{n}} + 1} \rightarrow \frac{1}{0+1} = 1$ $\boxed{\text{converges to } 1}$

[16] 9. Determine if the following series converge or diverge. If it is convergent, find its sum.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n}$. - Geometric series $a = -\frac{3}{4}$, $r = -\frac{3}{4}$

$\boxed{|r| < 1 - \text{Converges}}$

$S = \frac{a}{1-r} = \frac{-3/4}{1+3/4} = \frac{-3/4}{7/4} = \boxed{-\frac{3}{7}}$

(b) $\sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} + \frac{1}{3^n} \right)$ - next page

$$\sum \left(\frac{2}{n(n+1)} + \frac{1}{3^n} \right) = \underbrace{\sum \frac{2}{n(n+1)}}_{\uparrow \text{ Telescoping}} + \underbrace{\sum \frac{1}{3^n}}_{\uparrow \text{ Geometric}}$$

$$\frac{2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} .$$

$A(n+1) + Bn = 2$

$A + B = 0$

$A = 2$

So $B = -2$

$$= \frac{2}{n} - \frac{2}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = \sum \left(\frac{2}{n} - \frac{2}{n+1} \right) = \left(2 - \cancel{\frac{2}{2}}\right) + \left(\cancel{\frac{2}{2}} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \frac{2}{4}\right)$$

$$= 2$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \qquad r = \frac{1}{3},\ a = \frac{1}{3} \qquad \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$\boxed{\text{Sum} = 2 + \frac{1}{2} = \frac{5}{2}}$$