Abstract Algebra 2 **Spring 2007** **Solutions to HW #5**

1. (24,#12) Show that a group of order 56 has a proper, nontrivial normal subgroup.

   Proof: Let $|G| = 56 = 2^3 \cdot 7$. By the third Sylow Theorem, $n_7 \equiv 1 \bmod 7$ and divides 56. Thus the only possibilities are $n_7 = 1$ or $n_7 = 8$. However if $n_7 = 1$, then the Sylow 7-subgroup is normal and we are done. So it assume that it equals 8.

   Any Sylow 7-subgroup is cyclic and has 6 elements of order 7.The intersection of any two such groups consists of just the identity element. Thus there are $8 \cdot 6 = 48$ elements of order 7.

   Let H be a Sylow 2-subgroup. Then $|H| = 8$. None of these elements have order 7. Thus we now have $48 + 8 = 56 = |G|$ distinct elements. These are all the elements of the group. Therefore there is a unique Sylow 2-subgroup, and hence it is normal - done!

2. (24 #16) How many Sylow 5-subgroups of S_5 are there? Exhibit two.

   Proof: $|S_5| = 5! = 120 = 2^3 \cdot 3 \cdot 3 \cdot 5$. By the third Sylow Theorem the number of Sylow 5-subgroups must equal 1 mod 5 and divide 24. Hence $n_5 = 1$ or 6. So we will exhibit two such subgroups and then we are done. All such groups are cyclic, so they are generated by an element of order 5.

   Let

   $$H = \langle(1\ 2\ 3\ 4\ 5)\rangle = \{(1), (1\ 2\ 3\ 4\ 5), (1\ 3\ 5\ 2\ 4), (1\ 4\ 2\ 5\ 3), (1\ 5\ 4\ 3\ 2)\}.$$

   Then H is a Sylow 5-subgroup. If we find an element of order 5 in the group that is not in H, then it must generate a different Sylow 5-subgroup. In which case we would be done.

   Let $K = \langle(1\ 3\ 2\ 4\ 5)\rangle \neq H$.

3. (24, #18) Prove that a group of order 175 is abelian.

Proof: Note that $175 = 5^2 \cdot 7$. Since n_5 equals 1 mod 5 and divides 7, we have that $n_5 = 1$. Similarly, n_7 is one mod 7 and divides 25. By a little more checking we see that $n_7 = 1$. Let H be the Sylow 5-subgroup and K the Sylow 7-subgroup. Then $|H| = 5^2$, and so H is abelian. Since K has prime order, it clearly is also abelian. Since H is normal, the set HK is a subgroup of G. On the other hand since HK contains both H and K, HK has order bigger than 25. Since it also divides 175, it must have order 175, in other words $HK = G$. Finally we note that $H \cap K = \{e\}$, since the elements of H all have order a power of 5, while elements of K all have order 7 or 1. Thus by a Theorem $G \simeq H \times K$. Since both components are abelian, G is abelian.

4. (24, #26) Prove that a group of order 60 has exactly 4 elements of order 5 or exactly 24 elements of order 5.

Proof This is easy. $|G| = 60 = 2^2 \cdot 3 \cdot 5$. Thus $n_5 = 1$ or 6. Since any Sylow 5-subgroup consists of 4 elements of order 5 and the intersection of any two is trivial, the result follows.