Abstract Algebra 2 **Spring 2007** **Solutions to HW #4**

1. (7,#40) Let G be the group of rotations of a plane about a point P in the plane. Describe the orbit of a point Q in the plane.

   Proof: If $P = Q$, then the orbit of Q consist only of Q, since clearly any rotation about Q fixes Q.

   Suppose that $Q \neq P$. Let $g \in G$ be rotation through an angle of θ°. Then $g(Q)$ is clearly on the circle with center P and through the point Q. Since any point on this circle can be obtained from Q by rotation by an appropriate angle, it follows that the orbit is the circle with center at P and through Q.

2. (7 #42) This one was fairly easy, but almost impossible to write up on the typewriter, so I will not write it up

3. (24, #6) Exhibit a Sylow 2-subgroup of S_4. Describe an isomorphism from this group to D_4.

   Proof: Let H denote a Sylow 2-subgroup. Then $|H| = 8$. The subgroup H, if it is to be isomorphic to D_4 must have two elements of order 4, and five elements of order 2. To get an element of order 4 we need a 4-cycle say $\alpha = (1\ 2\ 3\ 4)$. Note that $\alpha^3 = (1\ 4\ 3\ 2)$. We also need an element of order 2. We have to be careful here. For note that if we use say $\beta = (1\ 2)$, then $\alpha\beta = (1\ 3\ 4)$, which has order 3 and this cannot be in H. However, if we use $\beta_1 = (1\ 3)$, then $\alpha\beta_1 = (1\ 4)(2\ 3)$. The difference is that 1 and 3 are not adjacent in α or α^3. Also $\beta_2 = (2\ 4)$ works for the same reason. To get 3 more elements of order 2, try the product of two disjoint 2-cycles, say $\beta_3 = (1\ 2)(3\ 4)$, $\beta_4 = (1\ 3)(2\ 4) = \alpha^2, \beta_5 = (1\ 4)(2\ 3)$. Once we through in the identity element, the set has 8 elements, all of whom have order a power of 2. There are two elements with order 4, and hence it has to be a Sylow 2-subgroup, if we believe that the subgroup will be isomorphic to D_4.

   To define a map to D_4, clearly we send

$$\alpha \mapsto R_{90}, \text{ and } \alpha^3 \mapsto R_{270}$$

   Thus $\beta_1\beta_2 = \beta_4 = \alpha^2 \mapsto R_{180}$. By checking the multiplication table for D_4 on page 33, we can send $\beta_1 \mapsto D$ and $\beta_2 \mapsto D'$.

We now have to decide where to send β_3 and β_5. We see that $VR_{90} = D'$. Since $\beta_3\alpha = \beta_2 \mapsto D'$ and $\alpha \mapsto R_{90}$ we must have $\beta_3 \mapsto V$. Finally $\beta_5 \mapsto V'$ the only element left.

4. Find all Sylow 3-subgroups of A_4

   Proof: $|A_4| = 4!/2 = 4 \cdot 3 = 12$. Thus a Sylow 3-subgroup has order 3, hence it is cyclic. The only elements in A_4 with order 3 are 3-cycles. All 3-cycles of S_4 are even permutations and hence are in A_4. Thus the subgroups are

   $$\langle(1\ 2\ 3)\rangle, \langle(1\ 2\ 4)\rangle, \langle(1\ 3\ 4)\rangle, \langle(2\ 3\ 4)\rangle$$

   Note that $(1\ 2\ 3)^2 = (1\ 3\ 2)$, so you have to be careful not to double count. There are 4 subgroups on the list and that is consistent with the Third Sylow Theorem.

5. (24, #10) Let H be a Sylow p-subgroup of G. Prove that H is the only p-subgroup of G contained in $N(H)$.

   Proof This is trivial with what I called Lemma B from the note. The Lemma says that any element in $N(H)$ of order a power of p is already in H. Since any element in a Sylow p-subgroup has order a power of p, we are done.