Abstract Algebra 2 **Spring 2007** **Solutions to HW #2**

1. Determine the minimal polynomial over $\mathbb{Q}$ for the element $1+i$.

   Proof: Complex roots to real polynomials come in conjugate pairs. So consider $f(x) = (x-(1+i))(x-(1-i)) = x^2-2x+2$. Clearly $1+i$ is a root of $f(x)$. Furthermore since it has degree 2 and since $1+i \notin \mathbb{Q}$, $f(x)$ has minimal degree.

2. Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ [One inclusion is obvious, for the other consider $(\sqrt{2}+\sqrt{3})^2$, etc.]. Conclude that $[\mathbb{Q}(\sqrt{2}+\sqrt{3}):\mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2}+\sqrt{3}$.

   Proof: (See Example 6 page 374 of book) Clearly $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$. Observe that $(\sqrt{2}+\sqrt{3})^{-1} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$. But
   $$(\sqrt{2}+\sqrt{3})^{-1} = \frac{1}{\sqrt{2}+\sqrt{3}} \cdot \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} = -(\sqrt{2}-\sqrt{3}).$$
   Thus $\sqrt{2}-\sqrt{3} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$. Hence $[(\sqrt{2}+\sqrt{3})+(\sqrt{2}-\sqrt{3})] = \sqrt{2} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$. Similarly, so is $\sqrt{3}$. Thus we have the reverse containment.

   One checks that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. (Actually, there is something to check here, namely that an element of the form $a+b\sqrt{2}$, $a,b \in \mathbb{Q}$ cannot be a square root of 3.) Thus $[\mathbb{Q}(\sqrt{3},\sqrt{2}):\mathbb{Q}] > 2$. On the other hand, since both $\sqrt{2}$ and $\sqrt{3}$ have degree 2 over $\mathbb{Q}$, it follows that $[\mathbb{Q}(\sqrt{3},\sqrt{2}):\mathbb{Q}] \leq 4$. However, it cannot equal 3, since degree of field extensions are multiplicative and 2 does not divide 3. Thus the degree equals 4.

   The irreducible polynomial satisfied by $\sqrt{2}+\sqrt{3}$ would clearly have to have degree 4, since $[\mathbb{Q}(\sqrt{2}+\sqrt{3}):\mathbb{Q}] = 4$. One checks that $\sqrt{2}+\sqrt{3}$ is a root of $f(x) = x^4-10x^2+1$. Since $f(x)$ has degree 4, it must be irreducible. For if it is not irreducible, then $\sqrt{2}+\sqrt{3}$ would be a root of one of its factors. This would contradict the fact that minimal polynomial of $\sqrt{2}+\sqrt{3}$ has degree 4.

3. Let $F = \mathbb{Q}(i)$. Prove that x^3-2 and x^3-3 are irreducible over F.

   Proof: Clearly $[\mathbb{Q}(i):\mathbb{Q}] = 2$, since i is a root of x^2+1. Suppose that $f(x) = x^3-2$ factored over $\mathbb{Q}(i)$. Then one of the terms would have degree 1. Thus $\mathbb{Q}(i)$ would contain a root a of $f(x)$, and so $\mathbb{Q}(a) \subseteq \mathbb{Q}(i)$. However, we know that $f(x)$ is irreducible over $\mathbb{Q}$. Thus $[\mathbb{Q}(a):\mathbb{Q}] = 3$, which contradicts the fact that $\mathbb{Q}(i)$ has degree 2 over $\mathbb{Q}$. The same proof works for x^3-2.

4. Let $f(x), g(x) \in \mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are relatively prime, show that they cannot have a common root in $\mathbb{C}$.

   Proof: Since $\mathbb{Q}[x]$ is a PID, and since $f(x)$ and $g(x)$ are relatively prime, there exists polynomials $h(x)$ and $k(x)$ in $\mathbb{Q}[x]$, such that
   $$h(x)f(x)+k(x)g(x) = 1.$$
   Suppose that $a \in \mathbb{C}$ is a common root. Then when we plug in a into the above equation we get
   $$h(a)f(a)+k(a)g(a) = 1$$
   But the left side of this equation is zero, since $f(a) = g(a) = 0$ - contradiction.