Math 494 Final - Take Home Spring 2007 Dr. J. Shapiro
Due Thursday, May 10, 3:00pm

1. Suppose that $f(x)$ and $g(x)$ are irreducible elements of $F[x]$ where F is a field, such that f and g are not associates of each other. Show that in any extension field E of F, there is no $a \in E$ that is a root of both $f(x)$ and $g(x)$. (Hint: Recall that $F[x]$ is a PID. Since f and g are irreducibles that are not associates, they are relatively prime. So what is the ideal (f, g)?)

2. Let $F = Z_2$ and let $f(x) = x^3 + x + 1 \in F[x]$. Suppose that a is a zero of $f(x)$ in some extension of F. How many elements does $F(a)$ have? Express each element of $F(a)$ in terms of a.

3. Without using the Primitive Element Theorem, prove that if $[K : F]$ is prime, then K has a primitive element.

4. Let E be an algebraic extension of a field F. If R is a ring such that $F \subseteq R \subseteq E$, show that R must be a field.

5. Find all Sylow 3-subgroups of A_4.

6. Prove that a group of order 135 has a subgroup of order 15.

7. Suppose that G is a finite group and all of its Sylow subgroups are normal. Show that G is the direct product of its Sylow subgroups.

8. Let SM denote the set of orientation preserving motions of the plane. Prove that SM is a normal subgroup of M. What is its index in M?

9. Prove that the map $M \longrightarrow \{1, r\}$ defined by $t_a\rho_\theta \mapsto 1$ and $t_a\rho_\theta r \mapsto r$ is a group homomorphism.

10. From the text, Chapter 31, # 22.