Exam # 2

Solution

1. Prove that if $x, y \in G$ are in the same conjugacy class, then $|C(x)| = |C(y)|$.

   Proof This is a trivial problem! I hope you appreciated it.

   We know that
   $$|C(x)| = \frac{|G|}{|cl(x)|} \text{ and } |C(y)| = \frac{|G|}{|cl(y)|}.$$
   By assumption, $cl(x) = cl(y)$. Since the denominators and numerators of the above fractions are equal, their quotients are equal. Done!

2. Let $|G| = p^2q^2$, p and q distinct primes such that $q \nmid p^2 - 1$, and $p \nmid q^2 - 1$. Prove that G is abelian.

   Proof Let H be s Sylow p-subgroup, so $|H| = p^2$. Then $n_p = 1 + kp$ and divides $|G| = p^2q^2$. Since n_p is relatively prime to p, it must divide q^2. The only divisors of q^2 are $1, q$ or q^2. We show that the last two cases are impossible. Suppose that $1 + kp = q$, then $kp = q - 1$. Hence p divides $q - 1$ and so p divides $q^2 - 1 = (q - 1)(q + 1)$, a contradiction with the assumption. If $n_p = 1 + kp = q^2$, then again we see that p divides $q^2 - 1$ - a contradiction. Thus $n_p = 1$. By a symmetric argument $n_q = 1$. Let K be the unique Sylow q-subgroup. Thus both H and K are normal subgroups of G. Hence HK is a subgroup of G. But $|HK|$ is divisible by both p^2 and q^2, hence $HK = G$.

   Note that every element of H (resp. K) has order a power p (resp. a power of q). Thus $H \cap K = \{e\}$. Therefore $G \cong H \times K$. Finally note that since both H and K have order the square of a prime number, each of them is abelian. Hence G is abelian.

3. If $H \lhd G$ and $|H| = p^k$, where p is prime, show that H is contained in every Sylow p-subgroup of G.

   Proof Since H has order a power of p, it is a subgroup of some Sylow p-subgroup K by the second Sylow Theorem. Let P be any other

Sylow p-subgroup. We have to show that $S \subseteq P$. By the third Sylow Thoerem, K and P are conjugates. Thus there exists $a \in G$ such that $aKa^{-1} = P$. Thus $aHa^{-1} \subseteq P$. But since H is normal in G, we have $aHa^{-1} = H$. Done!

4. Let P be a normal Sylow p-subgroup of G. Let H be a subgroup of G such that $p \mid |H|$. Show that $P \cap H$ is the unique Sylow p-subgroup of H.

   Proof We know that H contains a Sylow p-subgroup K. We have to show that $K = P \cap H$ and that K is the only Sylow p-subgroup of H. Clearly $|K| = p^t$. Thus by the second Sylow Theorem, $K \subseteq P$ and so $K \subseteq H \cap P$. On the other hand, $|H \cap P| = p^k$ for some k, since $H \cap P$ is a subgroup of P. Since K has order the maximal power of p that divides $|H|$, we see that $K = H \cap P$. But K was an arbitrary Sylow p-subgroup of H. Thus $K = H \cap P$ is the only Sylow p-subgroup of H.

5. Let $|G| = pqr$, where $p < q < r$ are prime numbers. Show that G contains a normal Sylow subgroup.

   Proof Assume the negative, namely G has no normal Sylow subgroups. Thus $n_r > 1$. Then $n_r = 1 + kr$ and divides pq. The only divisors of this last number that are bigger than 1 are p, q or pq. However n_r is bigger than q, thus it cannot equal either p or q. Thus $n_r = pq$. Notice that all the Sylow r-subgroups are cyclic of prime order. So the the intersection of any two consists of just the identity element. Since each has $r - 1$ elements of order q we see that there are $pq(r-1)$ elements of order r in G.

   Also $n_q > 1$. Then by an argument similar to the one above we have $n_q = r$ or pr. Pick the smaller of the two. Thus G contains at least $r(q-1)$ elements of order q.

   Finally we have $n_p > 1$. Thus $n_p = q, r$ or qr. Use the smallest of these numbers. Thus G contains at least $q(p-1)$ elements of order p.

   The above three sets are clearly disjoint, since they consists of elements of different prime orders. Thus G contains at least the sum of these numbers of elements. That is, G contains at least

$$pq(r-1) + r(q-1) + q(p-1) = pqr - pq + rq - r + qp - q$$

many elements. This reduces to $pqr + rq - r - q$. However it is straight forward to show that $rq > r + q$ (induction). Thus we have that G contains more than pqr elements - a contradiction. Thus one of n_r, n_q or n_p must equal one, which proves the result.

6. Let G be a group of order 56 and suppose that the Sylow 2-subgroup H is normal. Prove that $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. (Hint: Let P be a Sylow 7-subgroup. Then P acts on H via conjugation. Show that every element of H must have the same order. What is that order?)

   This problem was misstated, so I did not grade it. I meant to make the additional assumption that the Sylow 7-subgroup was not normal. As the problem is currently stated, it is clearly false (why?).

7. Let p be the smallest prime divisor of $|G|$. Show that any subgroup of G of index p is normal in G.

   Proof Let H be a subgroup of G of index p. Then there is a homomorphism from G to S_p whose kernel K is contained in H. Then by a standard argument $|G/K|$ must divide $|S_p| = p!$. Also $|G/K|$ divides $|G|$. Thus $|G/K| = pm$. If $m = 1$, then clearly $K = H$ and we are done. If $m > 1$, then m is divisible by a prime number at least as large as p. Thus $|G/K| = pqn$ where p and q are prime and $q \geq p$. Thus q divides $p!/p$. But this is impossible since $p!/p$ cannot be divisible by a prime number as big or bigger than p. Hence $m = 1$ and we are done.

8. Let G be a finite simple group that contains subgroups H and K such that $|G : H|$ and $|G : K|$ are prime numbers. Show that $|H| = |K|$.

   Proof If $|G : H| = |G : K|$, then $|H| = |K|$. Suppose false. Say $p = |G : H| < |G : K| = q$. Thus pq divides $|G|$. Furthermore, there is a homomorphism from G to S_p with kernel L contained in H. Thus q must divide $|G/L|$ and so q divides $|S_p| = p!$. But this is impossible since $p < q$. This contradiction proves the result.