Exam # 2

Due Friday, April 20 by 3pm

1. Prove that if $x, y \in G$ are in the same conjugacy class, then $|C(x)| = |C(y)|$.

2. Let $|G| = p^2q^2$, p and q distinct primes such that $q \nmid p^2 - 1$, and $p \nmid q^2 - 1$. Prove that G is abelian.

3. If $H \lhd G$ and $|H| = p^k$, where p is prime, show that H is contained in every Sylow p-subgroup of G.

4. Let P be a normal Sylow p-subgroup of G. Let H be a subgroup of G such that $p \mid |H|$. Show that $P \cap H$ is the unique Sylow p-subgroup of H.

5. Let $|G| = pqr$, where $p < q < r$ are prime numbers. Show that G contains a normal Sylow subgroup.

6. Let G be a group of order 56 and suppose that the Sylow 2-subgroup H is normal. Prove that $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. (Hint: Let P be a Sylow 7-subgroup. Then P acts on H via conjugation. Show that every element of H must have the same order. What is that order?)

7. Let p be the smallest prime divisor of $|G|$. Show that any subgroup of G of index p is normal in G.

8. Let G be a finite simple group that contains subgroups H and K such that $|G : H|$ and $|G : K|$ are prime numbers. Show that $|H| = |K|$.