KURZWEIL INTEGRAL REPRESENTATION OF INTERACTING PRANDTL-ISHLINSKII OPERATORS

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Abstract. We consider a system of operator equations involving play and Prandtl-Ishlinskii hysteresis operators. This system generalizes the classical mechanical models of elastoplasticity, friction and fatigue by introducing coupling between the operators. We show that under quite general assumptions the coupled system is equivalent to one effective Prandtl-Ishlinskii operator or, more precisely, to a discontinuous extension of the Prandtl-Ishlinskii operator based on the Kurzweil integral of the derivative of the state function. This effective operator is described constructively in terms of the parameters of the coupled system. Our result is based on a substitution formula which we prove for the Kurzweil integral of regulated functions integrated with respect to functions of bounded variation. This formula allows us to prove the composition rule for the generalized (discontinuous) Prandtl-Ishlinskii operators. The composition rule, which underpins the analysis of the coupled model, then establishes that a composition of generalized Prandtl-Ishlinskii operators is also a generalized Prandtl-Ishlinskii operator provided that a monotonicity condition is satisfied.

1. **Introduction.**

1.1. **Background and objectives.** Classical models of hysteresis effects are often defined as the superposition of a large number of simple hysteresis operators such as *plays*, *stops* or *non-ideal relays* [1, 2]. These simple hysteresis components model, for example, magnetic domains in a ferromagnetic material or individual fibers in an elastoplastic cable. In reality hysteretic components affect each other (via magnetic fields in magnetic media or friction forces in mechanical systems) but these interactions, by and large, are ignored in macroscopic models of hysteresis. The reason is that the inclusion of interactions usually leads to a dramatic complexification of the model and, specifically, makes identification of parameters a hard problem. For example, the Preisach model of ferromagnetic materials, which is obtained by superposition of non-interacting non-ideal relays, can be completely identified by measurements of the first and second-order hysteresis curves of the relationship between the magnetization and the applied field [3]. However, in another model of magnetic materials which is widely used in statistical physics, the Ising spin-interaction model where hysteresis emerges from the interaction of spins, any complete identification algorithm remains an open problem. Knowledge of the hysteresis curves up to any order k is not sufficient for recovering higher-order hysteresis curves or the adjacency matrix and spin switching thresholds [4].

Thus, the absence of interactions is often considered a necessary simplification for making the model tractable and efficient enough for engineering applications. However, adding even simple interactions via a mean-field term to the classical Preisach model (the *moving model* [5]) can account for the more complex hysteresis loops that are observed experimentally but forbidden by the Classical model. Furthermore, the rapid development of network science has stimulated greater interest in models that include interactions between hysteretic components. Interactions can introduce several new effects including a more complex hysteresis loop structure compared to standard models, as mentioned above. But perhaps the most interesting are cascading effects, also known as avalanches in discrete-state models, which allow perturbations to propagate through a network due to interactions between nodes [6]. In continuous models, this reveals itself as discontinuities in the input-output operator.

The objective of this work is to explore how the inclusion of interactions affects one classical model of hysteresis known as the Prandtl-Ishlinskii (PI) operator. To this end, we consider a network model that includes as particular cases: an extension of the PI model from a superposition of non-interacting elementary hysteresis operators called *plays* to a system of interacting plays; a discontinuous extension of the PI operator (we shall call it a generalized PI operator) that has properties similar to the continuous PI operator and for which the identification problem is completely solved; and a network with (generalized) PI operators at the nodes. Having introduced these classes of models (the most general being the network of PI operators), we attempt to obtain the solution operator for the network i.e. the operator that maps the time series of the input to the time series of the state of each node.

The PI operator is a phenomenological model of hysteretic relationships which has various physical interpretations. Examples include plasticity [7], friction[1] [8], fatigue accumulation [9], and complex constitutive laws coupling mechanical and electro-magnetic properties of smart materials [10]. The model admits effective numerical implementation. Furthermore, hysteresis loops generated by a PI operator

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[1]The Maxwell slip-friction model and the Prandtl-Ishlinskii model are mathematically identical.

have simple characteristic properties that can be used for solving the identification problem. And whenever empirical hysteresis loops exhibit (at least approximately) similar properties, or can be mapped to a set of loops with similar features by a coordinate transformation, a well-defined identification algorithm allows one to parameterize an approximating PI model. Essentially, in order to parameterize a PI operator, one needs to know one scalar function of a scalar variable called the *primary response* (PR) *curve*[2]. This curve is a plot of the output vs the input, which quantifies the response of the PI operator to monotonically increasing and decreasing inputs; experimentally, it is typically obtained by measuring the response to a slowly varying monotone input. The response of the PI operator to arbitrary piece-wise monotone inputs is then determined in terms of the PR curve by an explicit formula [11], which allows fast and accurate computation of the output for real time measurements of the input. This makes the PI operator a suitable model for designing model-based controllers [12]. A remarkable property of the PI operator, that is important in applications as well as being central to this work, is expressed by the *composition formula*. In short, a composition of PI operators is also a PI operator, and the PR function of the composition is the composition of the PR functions of the composed operators[3] [13]. In particular, the inverse of a PI operator is another PI operator. This property plays a key role in the compensation-based control of sensors, actuators and micro-electro-mechanical systems employing smart materials [14, 15].

In scalar plasticity theory the Prandtl-Ishlinskii model represents the constitutive relationship between variable stress (loading) and strain of an elasto-plastic material as a superposition of elementary constitutive laws known as elastic-ideal plastic elements (fibers), which are modeled by play operators [7]. The phenomenology of the PI model is based on the assumption that elementary fibers do not interact and respond to the external loading applied to all of them independently of each other. However, even in this classical setting, some interaction between the fibers can be expected to occur in practice A system of elastic and elastic-ideal plastic fibers which affect each other via frictional forces when stretched by external loading has been considered in [16]. That system, which assumes the Maxwell-slip model for the friction forces, is an example of the network models considered here. As was shown in [16], introducing interactions between plays into the PI model can result in two different scenarios. In the first, the system of interacting plays is equivalent to a single effective PI operator. That is, the interactions do not generate extra complexity in the input-output operator but merely change the parameters of the PI model. This will be the case when the interactions are sufficiently weak. In the second scenario, interactions produce complex hysteresis loops and ratcheting effect (non-closed loops) which cannot occur in a Prandtl-Ishlinskii model (nor in any model having return point memory, such as the Preisach and the Ising models). A detailed discussion of the mathematical properties of the PI operator and the mathematical theory of hysteresis operators can be found in [1, 11, 13, 17, 18].

In this work, we undertake a detailed analysis and rigorous mathematical underpinning of the first above scenario. For a mechanical application we refer the interested reader to [16]; the same paper also introduces a novel financial application where a network model with simple discontinuous generalized PI operators at

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[2]Also known as the *loading curve* in plasticity.

[3] We remark that a linear combination (weighted superposition) of PI operators is a PI operator too and follows immediately from the definition.

the nodes mimics the effect of momentum trading strategies on prices in a financial market.

1.2. **Methods and results.** Although systems (networks) of interacting PI operators have, to the best of our knowledge, not been studied before it is clear that the composition formula plays the main role in understanding and describing their behavior and in finding conditions which ensure that the whole system responds to inputs as one effective PI operator. Assuming that the input of the system is mapped to the time series of each component by an unknown PI operator, we obtain a system of operator equations where all of the terms are PI operators, some of them known (that is, their PR functions are given) and others unknown. These equations will contain compositions of known and unknown PI operators. If the composition formula works, then the PR functions of the unknown PI operators can be found from the algebraic system that is obtained from the system of operator equations by replacing each PI operator by the Nemytski operator generated by the PR function of the PI operator (equation (14)). After all the PR functions have been using this algebraic system (and thus all the PI operators have been identified) the argument is completed by verifying that the composition formula indeed works for all these PI operators. This requires one to verify that (a) all the solutions of the algebraic system (PR functions) are monotone, and (b) all the PR functions are continuous. If these two properties are confirmed, the above construction delivers an explicit solution of the system and ensures that each component's response to any external input is described by a PI operator, hence the hysteresis loops have a simple well-defined structure.

In this paper, we assume that the monotonicity condition (a) is satisfied (this corresponds to the first scenario that is realized for the mechanical and financial models mentioned above). However, in case of sufficiently strong interactions between the nodes, the network can produce a discontinuous response to continuous inputs. The question arises as to whether this discontinuous response is in any way similar to the response described by the PI operator in the continuous case.

Here we give a positive answer to this question by providing an effective solution to the system of interacting PI operators. Essentially, we generalize the above scheme based on the composition formula to the discontinuous case, thus lifting condition (b). To this end, we introduce an extension of the PI operator which can have a discontinuous PR function. This extension, called the generalized PI operator, is defined in terms of the Kurzweil integral of a regulated function representing the memory state of the PI operator with respect to a (discontinuous) PR function (Section 2).

Secondly, we prove the composition formula for the generalized PI operators. The composition formula states that the composition of two generalized PI operators is also a generalized PI operator provided a monotonicity condition is satisfied for one of them (Corollary 3.3). It allows us to apply the same scheme as was described above for the continuous setting to the discontinuous case. In other words, we show that a network of generalized PI operators is equivalent to one generalized PI operator if monotonicity holds (Theorem 3.1).

The proof of the composition formula is based on the substitution formula for the Kurzweil integral (Theorem 4.5) which, it appears, has not been proved before for regulated functions integrated with respect to a function of bounded variation — this is the result we need here. Hence, the proof of the substitution formula

presented below is of independent interest to Kurzweil integral theory (Section 4). Conclusions are presented at the end of the paper.

2. **Discontinuous extension of Prandtl-Ishlinskii operator.**

2.1. **Play operator.** Consider a simple hysteresis operator known in continuum mechanics as the *play operator* $\mathfrak{p}_r$ parameterized by $r > 0$. It was introduced in [1] first for continuous piecewise monotone inputs and was then extended to arbitrary continuous inputs from the dense set of piecewise monotone functions. Specifically, if $u \in C[0, T]$ is a given input function which is monotone (nondecreasing or non-increasing) in an interval $[t_0, t_1]$, and if the output $\xi_r(t_0) \in [u(t_0) - r, u(t_0) + r]$ is known, then we define the output $\xi_r(t)$ for $t \in [t_0, t_1]$ by

$$\xi_r(t) = \xi_r(t_0) + P_r(u(t) - \xi_r(t_0)), \tag{1}$$

where $P_r : \mathbb{R} \to \mathbb{R}$ is the *dead zone function*

$$P_r(x) = \max\{x - r, \min\{0, x + r\}\} \quad \text{for} \quad x \in \mathbb{R}. \tag{2}$$

Figure 1 presents the input-output diagram of the play operator and its mechanical interpretation.

The above definition was extended to regulated functions in [11]. We shall use the notation $\xi_r = \mathfrak{p}_r[\xi_r(0), u]$ for the play operator that maps the input time series $u = u(t)$ (in a mechanical setting, this is varying loading) to the output time series $\xi_r = \xi_r(t)$ (varying displacement) given an initial value $\xi_r(0)$. Note that this map is defined for any initial displacement from the interval $\xi_r(0) \in (u(0) - r, u(0) + r)$, see the legend of Figure 1.

Note that t is the time variable, while r can be interpreted as memory variable. All functions of t appearing in this text will be assumed to be *regulated and right continuous*. Recall that a function $f : [0, T] \to \mathbb{R}$ is regulated if both the left and the right limits $f(t-), f(t+)$ exist for each $t \in [0, T]$, with the convention $f(0-) = f(0)$, $f(T+) = f(T)$. The set of right continuous regulated functions is denoted by $G_R[0, T]$, and endowed with seminorms

$$\|f\|_{[t_1, t_2]} = \sup\{|f(t)| : t_1 \le t \le t_2\}, \tag{3}$$

and with the norm $\|f\|_{[0,T]}$ it is a Banach space. On the other hand, functions of r are also be regulated and may be right continuous or left continuous depending on the context. The set of left continuous regulated functions $\lambda : [0, K] \to \mathbb{R}$ will be denoted by $G_L[0, K]$.

2.2. **Prandtl-Ishlinskii model.** The Prandtl-Ishlinskii model is obtained by superposing play operators.

Figure 2(a) presents a schematic of a mechanical realization of the Prandtl-Ishlinskii model. For a system of N plays, the input-output relationship of this model is

$$\xi(t) = \sum_{i=1}^{N} \frac{1}{E_i} \mathfrak{p}_{r_i}[\xi_{r_i}(0), u](t), \tag{4}$$

where E_i is the spring stiffness and r_i is the maximal static friction force of the friction element for the i-th play. Usually, a continuous limit of this model is used (see, for example, [1, 11]), which has the form

$$\xi(t) = \int_0^\infty \mathfrak{p}_r[\xi_r(0), u](t) \, \mathrm{d}\mu(r). \tag{5}$$

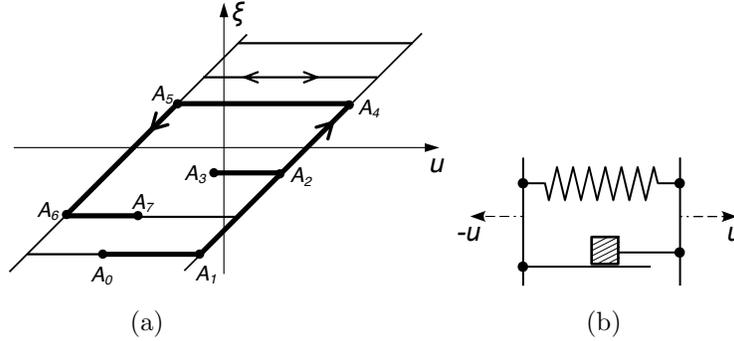(a)   (b)

FIGURE 1. (a) Input-output diagram of the play operator. The
point $(u(t), \xi(t))$ belongs to the band between the slanted lines $\xi = u + r$ and $\xi = u - r$ at all times. The point can move along the right
boundary of this band only upwards and along the left boundary
only downwards. Inside the band, it moves horizontally and can
move both left and right. The polyline $A_0 A_1 A_2 A_3 A_2 A_4 A_5 A_6 A_7$
shown in bold is an example of a trajectory of the point $(u(t), \xi(t))$.
Using the notation (u_i, ξ_i) for the coordinates of the point A_i, this
trajectory corresponds to the input $u(t)$ which increases from the
value u_0 to the value u_1 and further to the value u_2; then decreases
to the value u_3; then increases to the value u_4 through the value
u_2; then decreases to the value u_6 through the value u_5; and finally
increases to the value u_7. (b) Mechanical realization of the play:
an ideal spring with Young's modulus $E = 1$ and the Coulomb
friction element (object on a dry surface) connected in parallel.
The input $u(t)$ is the applied force (stress, loading). The output
$\xi(t)$ is the difference between the actual spring length and its rest
length (strain). When a force $u - E\xi$ applied to the object on
the dry surface is within the range $(-r, r)$ it is compensated by
the static friction force and the object remains stationary on the
surface i.e. the displacement ξ remains constant. When the friction
force reaches $\pm r$, a quasistatic motion begins. The kinetic friction
force is assumed to have the absolute value r equal to the maximal
value of the static friction. The balance of forces for the quasistatic
motion reads $u = E\xi \pm r$.

A convenient way to define the initial condition for the plays is to introduce the
memory state space

$$\Lambda = \{\lambda \in W_{loc}^{1,\infty}(0, \infty) : |\lambda'(r)| \leq 1 \text{ a.e. }\}, \tag{6}$$

and put

$$\xi_r(0) = \lambda(r) + P_r(u(0) - \lambda(r)) \tag{7}$$

for $\lambda \in \Lambda$. An initial condition (7) automatically satisfies the restriction $|\xi_r(0) - u(0)| \leq r$, which needs to be fulfilled for each play. We then consider the play
operator as a mapping which with a given memory state $\lambda \in \Lambda$ and a given input
$u \in G_R[0, T]$ associates the output $\xi_r \in G_R[0, T]$, and we write $\xi_r = \mathfrak{p}_r[\lambda, u]$.
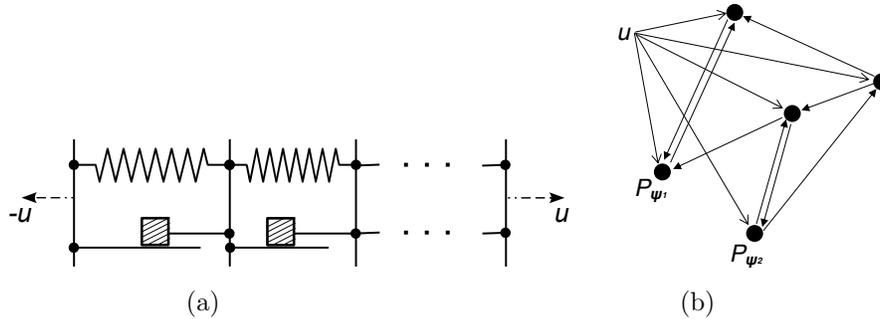
(a)   (b)

FIGURE 2. (a) Mechanical realization of the Prandtl-Ishlinskii
model: a consecutive connection of plays. The motion is qua-
sistatic. The applied force $u(t)$ is the same for all the plays. The
output $\xi(t)$ is the difference between the total length of the system
and its length at rest when all the spring forces are zero. Springs
have different stiffnesses as well as friction elements and so are char-
acterized by different maximal static friction values r_i. (b) Network
of Prandtl-Ishlinskii operators at the nodes with the input u.

In financial market modeling, the initial condition is typically chosen to be

$$\lambda_M(r) = \min\{-M + r, 0\}. \tag{8}$$

We shall see that such a choice also has advantages in that it simplifies the mathe-
matical formulas.

2.3. **Extension of the model via Kurzweil integral.** Let $\psi : [0, \infty) \to \mathbb{R}$
with $\psi(0) = 0$ be an arbitrary right continuous function with bounded variation.
Using the Kurzweil integral formalism, we define the *generalized Prandtl-Ishlinskii
operator* $\mathcal{P}_\psi : \Lambda \times G_R[0, T] \to G_R[0, T]$ generated by ψ by the formula

$$\mathcal{P}_\psi[\lambda, u](t) = -\int_0^\infty \frac{\partial-}{\partial r} \mathfrak{p}_r[\lambda, u](t) \, \mathrm{d}\psi(r), \tag{9}$$

where the symbol $\partial-/\partial r$ denotes the left partial derivative with respect to r. The
function ψ is called the *primary response curve* or the *loading curve* of $\mathcal{P}_\psi$.

To ensure that this definition is both meaningful and compatible with the stan-
dard definition (5) of the Prandtl-Ishlinskii operator, we restrict the admissible
domain Λ of memory configurations λ by choosing a fixed number $K > 0$ and
putting

$$\Lambda_K = \{\lambda \in \Lambda : \lambda(r) = 0 \text{ for } r \geq K, \ \lambda'|_{[0,K]} \in G_L[0, K]\}. \tag{10}$$

By [11, Propositions 2.7.5, 2.7.6], if $|u(t)| \leq U$ for every $t \in [0, T]$, then $\mathfrak{p}_r[\lambda, u](t) =$
0 for all $r \geq \max\{K, U\}$, hence in (9) we integrate over a bounded interval. In
particular, λ_M from (8) belongs to Λ_M. Furthermore, still using [11, Propositions
2.7.5, 2.7.6], the function $r \mapsto \frac{\partial-}{\partial r} \mathfrak{p}_r[\lambda, u](t)$ takes only values ± 1 in $(0, U)$ with
finitely many jumps in each interval $[a, U]$ with $a > 0$, hence it belongs to $G_L[a, R]$
for every $0 < a < R < \infty$. Then formula (9) can be obtained from (5) using
integration by parts. In this case,

$$\psi(r) = \int_0^r (r - s) \, \mathrm{d}\mu(s)$$

and so the loading curve ψ is continuous. This function has a clear meaning: if $\lambda = 0$ (that is, all the forces at the initial moment are zero) and the input u is increasing, then the output of the Prandtl-Ishlinskii model is $\xi(t) = \psi(u(t))$.

It is important to remark that both the function ψ and the integrand in the Kurzweil integral definition (9) of the generalized Prandtl-Ishlinskii operator may be discontinuous. Also, the play $\mathfrak{p}_{r_0}$ with threshold r_0 can be considered as a special case of the Prandtl-Ishlinskii operator with the choice $\psi(r) = (r - r_0)^+$.

3. **Main results.**

3.1. **Problem statement and main theorem.** In this section, we develop a method for solving systems of the form

$$\mathbf{w}(t) = \boldsymbol{\mathcal{P}}[\mathbf{w}](t) + \boldsymbol{c}\, u(t), \tag{11}$$

where $u \in G_R[0,T]$ is a given right continuous regulated input function, $\boldsymbol{c} = (c_1, \ldots, c_n)$ is a given constant vector, $\boldsymbol{\mathcal{P}}$ is a vector operator of the form

$$(\boldsymbol{\mathcal{P}}(\mathbf{w}))_i(t) = \sum_{j=1}^{n} \mathcal{P}_{\psi_{ij}}[\lambda_{ij}, w_j](t)\,, \tag{12}$$

$\mathcal{P}_{\psi_{ij}}$ is a generalized Prandtl-Ishlinskii operator with a primary response curve ψ_{ij}, and with initial condition $\lambda_{ij} \in \Lambda_K$. The unknown in the problem is the function $\mathbf{w}(t) = (w_1(t), w_2(t), \ldots, w_n(t))$.

The form of (11) that is most relevant here is the system

$$w_i(t) = \sum_{j=1}^{n} a_{ij} \mathcal{P}_{\psi_j}[\lambda_j, w_j](t) + c_i u(t)\,, \qquad i = 1, \ldots, n, \tag{13}$$

which describes a network of n generalized Prandtl-Ishlinskii operators $\mathcal{P}_{\psi_i}[\lambda_i, \cdot]$, see Figure 2(b). Here $(a_{ij})_{i,j=1,\ldots,n}$ is the adjacency matrix of the network so that two nodes are connected if $a_{ij} \neq 0$ and the value of a_{ij} measures the strength of their interaction. The network is driven by an external input u. From (13), the input w_i of the i-th node is a weighted sum of the outputs $\mathcal{P}_{\psi_j}[\lambda_j, w_j]$ of all the nodes connected to it and the external input u. Examples of mechanical systems with friction and plastic elements leading to model (13), as well as a financial application, can be found in [16].

The main result of this paper is the following theorem.

Theorem 3.1. *Assume that primary response functions ψ_{ij} of the operators $\mathcal{P}_{\psi_{ij}}$ are right continuous, have bounded variation, and satisfy $\psi_{ij}(0) = 0$. If the algebraic system*

$$\varphi_i(r) = \sum_{j=1}^{n} \psi_{ij}(\varphi_j(r)) + c_i r, \qquad i = 1, \ldots, n, \tag{14}$$

admits a nondecreasing right continuous solution $(\varphi_1(r), \ldots, \varphi_n(r))$ for all $r \geq 0$ with $\varphi_i(0) = 0$ then, for a suitable choice of initial states λ_{ij}, $\lambda^i \in \Lambda_K$, problem (11)–(12) has a solution $w_i = \mathcal{P}_{\varphi_i}[\lambda^i, u]$ in the class of generalized Prandtl-Ishlinskii operators for all regulated inputs u.

We will see below in Corollary 3.4 how the initial conditions have to be chosen for the validity of the superposition formula. In addition, the condition that (14) admits nondecreasing solutions is quite restrictive. Even if all the ψ_{ij} are linear, that is, $\psi_{ij}(z) = a_{ij} z$ with $a_{ij} > 0$, the condition may fail to hold. On the other

hand, if right-continuous functions ψ_{ij} are non-decreasing and bounded, then the existence of nondecreasing solutions φ_i follows from a variant of the Birkhoff-Tarski fixed point theorem with compactness based on the Helly selection principle (the financial model in [16] satisfies these properties). Generically, continuity of functions ψ_{ij} (which is equivalent to continuity of the operators $\mathcal{P}_{\psi_{ij}}$ in (12)) does not guarantee continuity of the solution operators $w_i = \mathcal{P}_{\varphi_i}[\lambda^i, u]$. Interaction of the nodes can naturally result in discontinuity of operators $\mathcal{P}_{\varphi_i}$. This is part of the reason for introducing the generalized Prandtl-Ishlinskii operator. Indeed, the simplest equation

$$w(t) = \mathcal{P}_\psi[\lambda, w](t) + u(t)$$

with $n = 1$ admits a continuous solution operator $w = (I - \mathcal{P}_\psi)^{-1}u$ if and only if the function $r \to r - \psi(r)$ admits a continuous increasing inverse. However, additional assumptions can ensure uniqueness and continuity of the Prandtl-Ishlinskii solution operators $w_i = \mathcal{P}_{\varphi_i}[\lambda^i, u]$, $i = 1, \ldots, n$ for problem (11)–(12). For example, this is the case for any system (13) with smooth functions ψ_i and sufficiently small coefficients a_{ij} (that is, a network with weak interaction; see [16] for a mechanical example). Strong interactions typically result in discontinuities which represent cascading effects (avalanches) in the network.

3.2. **Composition formula.** The main step in proving Theorem 3.1 consists of generalizing the Prandtl-Ishlinskii composition formula from [13] to the case of discontinuous primary response curves.

Proposition 3.2. *Let $\lambda \in \Lambda_K$, $u \in G_R[0, T]$ be given, and let $\varphi : [0, \infty) \to [0, \infty)$ be a nondecreasing function, $\varphi(0) = 0$, $\varphi(\infty) =: \varphi_\infty \leq \infty$. Let φ^{-1} be the left continuous inverse of φ, that is,*

$$\varphi^{-1}(s) = \inf\{t \geq 0 : s \leq \varphi(t)\}. \tag{15}$$

For $s \geq 0$ put

$$\lambda_\varphi(s) = -\int_s^{\varphi_\infty} \lambda'(\varphi^{-1}(r))\,\mathrm{d}r\,. \tag{16}$$

Set $v(t) = \mathcal{P}_\varphi[\lambda, u](t)$. Then $\lambda_\varphi \in \Lambda_{\varphi(K)}$, $v \in G_R[0, T]$ and for all $s \geq 0$,

$$\mathfrak{p}_s[\lambda_\varphi, v](t) = -\int_s^{\varphi_\infty} \frac{\partial-}{\partial r}\big(\mathfrak{p}_r[\lambda, u](t)\big)\Big|_{r=\varphi^{-1}(\varrho)}\,\mathrm{d}\varrho\,. \tag{17}$$

Proof. It is enough to prove formula (17) for one memory update, that is for the input $u(t) = \lambda(0) + (\hat{u} - \lambda(0))H(t - \tau)$ where H is the Heaviside function and $t > \tau \geq 0$, and $\hat{u} \in \mathbb{R}$. To be definite, assume that $\hat{u} > \lambda(0)$. The function

$$\mu(r) = \begin{cases} \hat{u} - r & \text{for } r \in [0, r_u), \\ \lambda(r) & \text{for } r \geq r_u, \end{cases} \tag{18}$$

where r_u is the smallest root of the equation $r + \lambda(r) = \hat{u}$, then represents the updated value of the play $\mathfrak{p}_r[\lambda, u]$ at the moment t and we indeed have $\mu \in \Lambda_{\max\{\hat{u}, K\}}$. By Theorem 4.5, the updated value of v at the moment t is given by the formula

$$v(t) = -\int_0^\infty \mu'(r)\,\mathrm{d}\varphi(r) = -\int_0^{\varphi_\infty} \mu'(\varphi^{-1}(\varrho))\,\mathrm{d}\varrho.$$

We now define μ_φ as the right hand side in (17), that is,

$$\mu_\varphi(s) = -\int_s^{\varphi_\infty} \mu'(\varphi^{-1}(\varrho))\,\mathrm{d}\varrho.$$

Using (18), we obtain

$$\mu_\varphi(s) = \begin{cases} v(t) - s & \text{for } s \in [0, \varphi(r_u-)), \\ \lambda_\varphi(s) & \text{for } s \geq \varphi(r_u-), \end{cases} \tag{19}$$

and it is easily checked that $s = \varphi(r_u-)$ is the smallest root of the equation $s + \lambda_\varphi(s) = v(t)$. Hence μ_φ is the update of the play applied to v with initial memory λ_φ, which is what we required. □

From Proposition 3.2, we obtain the following composition formula for generalized Prandtl-Ishlinskii operators.

Corollary 3.3. *Let* $\lambda \in \Lambda_K$, $u \in G_R[0, T]$, φ, *and* v *be as in Proposition* 3.2, *and let* $\psi : [0, \infty) \to \mathbb{R}$ *with* $\psi(0) = 0$ *be a right continuous function of bounded variation. Then*

$$\mathcal{P}_\psi[\lambda_\varphi, v] = \mathcal{P}_{\psi \circ \varphi}[\lambda, u]. \tag{20}$$

Proof. Note first that by virtue of (17) we have

$$\frac{\partial-}{\partial s}\big(\mathfrak{p}_s[\lambda_\varphi, v](t)\big) = \frac{\partial-}{\partial r}\big(\mathfrak{p}_r[\lambda, u](t)\big)\Big|_{r=\varphi^{-1}(s)}.$$

Hence, by Theorem 4.5 (see below),

$$\mathcal{P}_\psi[\lambda_\varphi, v] = -\int_0^\infty \frac{\partial-}{\partial s}\big(\mathfrak{p}_s[\lambda_\varphi, v](t)\big)\, \mathrm{d}\psi(s) = -\int_0^\infty \frac{\partial-}{\partial r}\big(\mathfrak{p}_r[\lambda, u](t)\big)\, \mathrm{d}(\psi \circ \varphi)(r),$$

and the assertion follows. □

Consider now nondecreasing functions $\varphi_i : [0, \infty) \to [0, \infty)$, $\varphi_i(0) = 0$, $i = 1, \ldots, n$, and another system $\psi_{ij} : [0, \infty) \to \mathbb{R}$, $i, j = 1, \ldots, n$, of right continuous functions of bounded variation with $\psi_{ij}(0) = 0$, as in Theorem 3.1. Corollary 3.3 immediately implies the following statement.

Corollary 3.4. *Assume that relations* (14) *hold for all* $r \geq 0$. *For* $\lambda^i \in \Lambda_K$ *and* $u \in G_R[0, T]$ *set* $w_i = \mathcal{P}_{\varphi_i}[\lambda^i, u]$, $i = 1, \ldots, n$. *Then*

$$w_i(t) = \sum_{j=1}^n \mathcal{P}_{\psi_{ij}}[\lambda^i_{\varphi_j}, w_j](t) + c_i u(t) \tag{21}$$

for all $t \in [0, T]$ *and* $i = 1, \ldots, n$, *with* $\lambda^i_{\varphi_j}$ *related to* λ^i *and* φ_j *as in* (16).

Theorem 3.1 now follows from Corollaries 3.3 and 3.4. In particular, if $\lambda^i(r) = \min\{-M_i + r, 0\}$ as in (8), then $\lambda^i_{\varphi_j}(r) = \min\{-\varphi_j(M_i-) + r, 0\}$ for each $i, j = 1, \ldots, n$.

From their definition, the class of generalized Prandtl-Ishlinskii operators is closed with respect to linear superposition (parallel connections). From Corollary 3.3 it follows that the class of generalized Prandtl-Ishlinskii operators with increasing primary response curves is also closed with respect to the composition operation (cascade connections). Finally, Theorem 3.1 implies that any system (13) (network connections) of the generalized Prandtl-Ishlinskii operators with increasing bounded primary response curves and a non-negative adjacency matrix $(a_{ij})_{i,j=1,\ldots,n}$ is equivalent to a set of independent generalized Prandtl-Ishlinskii operators.

It is standard to define the output $\xi = \xi(t)$ of a network model as a weighted sum of the states of its nodes, that is, for system (13) to set

$$\xi(t) = \sum_{j=1}^{n} \mu_j \mathcal{P}_{\psi_j}[\lambda_j, w_j](t) \tag{22}$$

(cf. (4), (5)). If the conditions of Theorem 3.1 are satisfied, then the input-output operator mapping the input u of the network (13) to its output ξ is a generalized Prandtl-Ishlinskii operator $\xi(t) = \mathcal{P}_\phi[\lambda, u](t)$ with $\phi(r) = \sum_{i=1}^{n} \mu_i \psi_i(\varphi_i(r))$.

4. **Substitution in Kurzweil integrals.** In this section, we recall the definition and some basic properties of the Kurzweil integral — introduced in [19] as a framework for solving ODEs with singular right-hand sides. We cite most of the results without proof. The interested reader can find further details in [20, 21, 22]. However, Theorem 4.5, which plays an important role here, appears to be new and its detailed proof is given at the end of this section.

The original definition in [19] is not suitable for the integral formulation of discontinuous evolutionary variational inequalities, and this is why the Young integral was used instead in [23]. However, the extension of the Kurzweil integral in [20] contains the Young integral as special case whilst preserving the advantage of the easier Kurzweil formalism. Here, we only deal with right-continuous evolution processes and Definition 4.1 below, which dates back to [21], will turn out to be sufficient for our purposes.

The basic concept in Kurzweil integration theory is that of a δ-*fine partition*. Consider a nondegenerate closed interval $[a, b] \subset \mathbb{R}$, and denote by $\mathcal{D}_{a,b}$ the set of all divisions of the form

$$d = \{t_0, \ldots, t_m\}, \quad a = t_0 < t_1 < \cdots < t_m = b. \tag{23}$$

With a division $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ we associate *partitions* D defined as

$$D = \{(\varrho_j, [t_{j-1}, t_j]) ; j = 1, \ldots, m\}; \qquad \varrho_j \in [t_{j-1}, t_j] \quad \forall j = 1, \ldots, m. \tag{24}$$

We define the set

$$\Gamma(a, b) := \{\delta : [a, b] \to \mathbb{R} ; \delta(t) > 0 \quad \text{for every} \quad t \in [a, b]\}. \tag{25}$$

An element $\delta \in \Gamma(a, b)$ is called a *gauge*. For $t \in [a, b]$ and $\delta \in \Gamma(a, b)$ we denote

$$I_\delta(t) := (t - \delta(t), t + \delta(t)). \tag{26}$$

Definition 4.1. Let $\delta \in \Gamma(a, b)$ be a given gauge. A partition D of the form (24) is said to be δ-*fine* if for every $j = 1, \ldots, m$ we have

$$\varrho_j \in [t_{j-1}, t_j] \subset I_\delta(\varrho_j),$$

and the following implications hold:

$$\varrho_j = t_{j-1} \Rightarrow j = 1, \quad \varrho_j = t_j \Rightarrow j = m.$$

The set of all δ-fine partitions is denoted by $\mathcal{F}_\delta(a, b)$.

It is easy to see that $\mathcal{F}_\delta(a, b)$ is nonempty for every $\delta \in \Gamma(a, b)$; this follows e. g. from [24, Lemma 1.2].

For given functions $f, g : [a, b] \to \mathbb{R}$ and a partition D of the form (24) we define the Kurzweil integral sum $K_D(f, g)$ by the formula

$$K_D(f, g) = \sum_{j=1}^{m} f(\varrho_j) (g(t_j) - g(t_{j-1})). \tag{27}$$

Definition 4.2. Let $f, g : [a, b] \to \mathbb{R}$ be given. We say that $J \in \mathbb{R}$ ($J^* \in \mathbb{R}$) is the *Kurzweil integral* over $[a, b]$ of f with respect to g and denote

$$J = \int_a^b f(t) \, \mathrm{d}g(t), \tag{28}$$

if for every $\varepsilon > 0$ there exists $\delta \in \Gamma(a, b)$ such that for every $D \in \mathcal{F}_\delta(a, b)$ we have

$$|J - K_D(f, g)| \leq \varepsilon. \tag{29}$$

Using the fact that the implication

$$\delta \leq \min\{\delta_1, \delta_2\} \quad \Rightarrow \quad \mathcal{F}_\delta(a, b) \subset \mathcal{F}_{\delta_1}(a, b) \cap \mathcal{F}_{\delta_2}(a, b) \tag{30}$$

holds for every $\delta, \delta_1, \delta_2 \in \Gamma(a, b)$ it is easily verified that the value of J in Definition 4.2 is uniquely defined.

The Kurzweil integral satisfies most of the standard integral properties concerning additivity and linearity with respect to f and g.

Clearly, the set of discontinuity points of every regulated function is at most countable. Following [22], we denote by $G(a, b)$ the set of all regulated functions $f : [a, b] \to \mathbb{R}$. Let us introduce in $G(a, b)$ a system of seminorms

$$\|f\|_{[s,t]} := \sup\{|f(\tau)| \, ; \, \tau \in [s, t]\} \tag{31}$$

for any subinterval $[s, t] \subset [a, b]$. Indeed, $\|\cdot\|_{[a,b]}$ is a norm and with this norm, $G(a, b)$ becomes a Banach space. Let us note that the space $C[a, b]$ of *continuous functions* $f : [a, b] \to \mathbb{R}$ is a closed subspace of $G(a, b)$ with respect to the norm $\|\cdot\|_{[a,b]}$. Moreover, every regulated function can be uniformly approximated by *step functions* of the form

$$w(t) = \sum_{k=0}^m \hat{c}_k \chi_{\{t_k\}}(t) + \sum_{k=1}^m c_k \chi_{(t_{k-1}, t_k)}(t), \quad t \in [a, b], \tag{32}$$

where $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ is a division, χ_A denotes the characteristic function of a set $A \subset [0, b]$ (that is, $\chi_A(t) = 1$ if $t \in A$, $\chi_A(t) = 0$ if $t \notin A$), and $\hat{c}_0, \ldots, \hat{c}_m, c_1, \ldots, c_m$ are real numbers. We see in particular that the space $BV(a, b)$ of all functions of bounded variation on $[a, b]$ is contained as a dense subset in $G(a, b)$. In the next section, we will restrict ourselves to the spaces $G_R(a, b)$, $BV_R(a, b)$ of right-continuous functions from $G(a, b)$, $BV(a, b)$, respectively.

Remark 1. *The additivity property of the Kurzweil integral needs some comment. Whenever we integrate functions f, g defined in $[a, b]$ over an interval $[r, s] \subset [a, b]$, we implicitly consider their restrictions $f|_{[r,s]}, g|_{[r,s]}$. In particular, for regulated functions, we have e. g. $f|_{[r,s]}(s+) = f(s)$, $f|_{[r,s]}(r-) = f(r)$.*

Note that we deal here with functions that are defined for all $t \in [a, b]$. The concept of "almost everywhere" is meaningless.

The following explicit formulas can easily be derived from the definition.

Proposition 4.3. *For every $f : [a, b] \to \mathbb{R}$, $g \in G(a, b)$, $a \leq r \leq b$, we have*

(i) $\displaystyle \int_a^b \chi_{\{r\}}(t) \, \mathrm{d}g(t) = g(r+) - g(r-)$,

(ii) $\displaystyle \int_a^b f(t) \, \mathrm{d}\left(\chi_{\{r\}}\right)(t) = \begin{cases} 0 & \text{if } r \in (a, b) \\ -f(a) & \text{if } r = a, \\ f(b) & \text{if } r = b, \end{cases}$

(iii) $\displaystyle\int_a^b \chi_{(r,s)}(t)\,\mathrm{d}g(t) \;=\; g(s-) - g(r+) \quad \forall\, s \in (r,b]\;,$

(iv) $\displaystyle\int_a^b f(t)\,\mathrm{d}\left(\chi_{(r,s)}\right)(t) \;=\; f(r) - f(s) \quad \forall\, s \in (r,b]\;.$

We see in particular that the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists whenever one of the functions f, g is a step function and the other one is regulated. By a density argument, we have the following classical result, see [22].

Theorem 4.4. (Properties of the Kurzweil integral)

(i) *If $f \in G(a,b)$ and $g \in BV(a,b)$, then $\int_a^b f(t)\,\mathrm{d}g(t)$ exists and satisfies the estimate*

$$\left| \int_a^b f(t)\,\mathrm{d}g(t) \right| \;\leq\; \|f\|_{[a,b]} \operatorname*{Var}_{[a,b]} g\,. \tag{33}$$

(ii) *If $f \in BV(a,b)$ and $g \in G(a,b)$, then $\int_a^b f(t)\,\mathrm{d}g(t)$ exists and satisfies the estimate*

$$\left| f(a)\,g(a) + \int_a^b f(t)\,\mathrm{d}g(t) \right| \;\leq\; \left(|f(b)| + \operatorname*{Var}_{[a,b]} f \right) \|g\|_{[a,b]}\,. \tag{34}$$

(iii) *For every $f \in G(a,b)$, $g \in BV(a,b)$ we have the integration-by-parts formula*

$$\int_a^b f(t)\,\mathrm{d}g(t) + \int_a^b g(t)\,\mathrm{d}f(t) \;=\; f(b)\,g(b) - f(a)\,g(a) \tag{35}$$

$$+ \sum_{t \in [a,b]} \Big((f(t) - f(t-))\,(g(t) - g(t-)) - (f(t+) - f(t))\,(g(t+) - g(t)) \Big)\,.$$

(iv) *If $f_n \in G(a,b)$ and $g_n \in BV(a,b)$ are such that $\|f_n - f\|_{[a,b]}$, $\|g_n - g\|_{[a,b]} \to 0$ as $n \to \infty$, and $\operatorname{Var}_{[a,b]} g_n \leq C$ independently of n, then*

$$\lim_{n \to \infty} \int_a^b f_n(t)\,\mathrm{d}g_n(t) \;=\; \int_a^b f(t)\,\mathrm{d}g(t)\,. \tag{36}$$

Our main result for the Kurzweil integral is the following substitution formula.

Theorem 4.5. *Let $f : [0,b] \to \mathbb{R}$ be a bounded function such that $f\big|_{[a,b]} \in G(a,b)$ for all $a \in (0,b)$. Let $\varphi : [0,b] \to [0,B]$ be a nondecreasing function, $\varphi(0) = 0$, $\varphi(b) = B$, and let $\psi : [0,B] \to \mathbb{R}$ be a right continuous function of bounded variation. Let $\varphi^{-1} : [0,B] \to [0,b]$ be as in (15). Then for all $a \in [0,b)$ we have*

$$\int_a^b f(t)\,\mathrm{d}(\psi \circ \varphi)(t) = \int_{\varphi(a)}^{\varphi(b)} f(\varphi^{-1}(s))\,\mathrm{d}\psi(s)\,. \tag{37}$$

The proof is divided into several steps.

Lemma 4.6. *The function φ^{-1} defined by (15) is left continuous and, for all $a \leq c < d \leq b$, the following implications hold.*

(i) $\varphi^{-1}(s) = c \;\Rightarrow\; s \in [\varphi(c-), \varphi(c+)]$;
(ii) $s \in (\varphi(c-), \varphi(c+)] \;\Rightarrow\; \varphi^{-1}(s) = c$;
(iii) $\varphi^{-1}(s) \in (c,d) \;\Rightarrow\; s \in (\varphi(c+), \varphi(d-)]$;
(iv) $s \in (\varphi(c+), \varphi(d-)) \;\Rightarrow\; \varphi^{-1}(s) \in (c,d)$.

Proof of Lemma 4.6. The left continuity of φ^{-1} is a consequence of the following series of implications for all r, s:

$r < \varphi^{-1}(s) \ \Rightarrow \ \varphi(r) < s \ \Rightarrow \ \exists \delta > 0 : \varphi(r) < s - \delta \ \Rightarrow \ \exists \delta > 0 : \varphi^{-1}(s - \delta) \geq r \ \Rightarrow \ \varphi^{-1}(s-) \geq r.$

We prove (i)–(iv) by four similar series of implications:

$$s < \varphi(r-) \ \Rightarrow \ \exists \delta > 0 : s < \varphi(r - \delta) \ \Rightarrow \ \exists \delta > 0 : \varphi^{-1}(s) \leq r - \delta \ \Rightarrow \ \varphi^{-1}(s) < r, \tag{38}$$

$$s > \varphi(r+) \ \Rightarrow \ \exists \delta > 0 : s > \varphi(r + \delta) \ \Rightarrow \ \exists \delta > 0 : \varphi^{-1}(s) \geq r + \delta \ \Rightarrow \ \varphi^{-1}(s) > r, \tag{39}$$

$$r < \varphi^{-1}(s) \ \Rightarrow \ \exists \delta > 0 : r + \delta < \varphi^{-1}(s) \ \Rightarrow \ \exists \delta > 0 : \varphi(r + \delta) < s \ \Rightarrow \ \varphi(r+) < s, \tag{40}$$

$$r > \varphi^{-1}(s) \ \Rightarrow \ \exists \delta > 0 : r - \delta > \varphi^{-1}(s) \ \Rightarrow \ \exists \delta > 0 : \varphi(r - \delta) \geq s \ \Rightarrow \ \varphi(r-) \geq s. \tag{41}$$

Then (i) and (iv) follow from (38)–(39), (ii) and (iii) follow from (40)–(41). □

Lemma 4.7. *Let φ be as in Theorem 4.5, and let $\psi : [0, B] \to [0, C]$ be a nondecreasing right continuous function, $\psi(0) = 0$, $\psi(B) = C$. Let ψ^{-1}, $(\psi \circ \varphi)^{-1}$ be defined as in (15). Then for every $r \in [0, C]$ we have $(\psi \circ \varphi)^{-1}(r) = \varphi^{-1}(\psi^{-1}(r))$.*

Proof of Lemma 4.7. Let $r \in [0, C]$ be given. Put $p = (\psi \circ \varphi)^{-1}(r)$, $q = \varphi^{-1}(\psi^{-1}(r))$. Assume first that $p > q$. For $t \in (q, p)$ we then have $\varphi(t) \geq \psi^{-1}(r)$, $\psi(\varphi(t)) < r$. Then $s = \varphi(t)$ satisfies $\psi(s) < r$ and $s \geq \psi^{-1}(r)$. The latter inequality implies $\psi(s+) \geq r$, which contradicts the right continuity of ψ.

Similarly, assuming $p < q$, we have for $t \in (p, q)$ that $\varphi(t) < \psi^{-1}(r)$, $\psi(\varphi(t)) \geq r$, which is a contradiction. □

The formula in Lemma 4.7 does not hold in general if ψ is not right continuous. Consider the example

$$\varphi(t) = \psi(t) = \begin{cases} 0 & \text{for } t \in [0, 1], \\ 1 & \text{for } t \in (1, 2), \\ 2 & \text{for } t = 2. \end{cases}$$

Then

$$(\psi \circ \varphi)(t) = \begin{cases} 0 & \text{for } t \in [0, 2), \\ 2 & \text{for } t = 2, \end{cases}$$

while $\varphi^{-1}(\psi^{-1}(t)) = 1$ for $t \in (0, 1)$.

Lemma 4.8. *Let the hypotheses of Theorem 4.5 hold. Then for all $a \in (0, b)$ we have*

$$\int_a^b f(t) \, \mathrm{d}\varphi(t) = \int_{\varphi(a)}^{\varphi(b)} f(\varphi^{-1}(s)) \, \mathrm{d}s. \tag{42}$$

Proof of Lemma 4.8. For functions f of the form $f(t) = \chi_{\{c\}}(t)$ or $f(t) = \chi_{(c,d)}(t)$, we have the Kurzweil integration formulas

$$\int_a^b \chi_{\{c\}}(t) \, \mathrm{d}\varphi(t) = \varphi(c+) - \varphi(c-), \qquad \int_a^b \chi_{(c,d)}(t) \, \mathrm{d}\varphi(t) = \varphi(d-) - \varphi(c+).$$

On the other hand, by Lemma 4.6, we have $\chi_{\{c\}}(\varphi^{-1}(s)) = \chi_{A_c}(s)$ with

$$(\varphi(c-), \varphi(c+)] \subset A_c \subset [\varphi(c-), \varphi(c+)],$$

and $\chi_{(c,d)}(\varphi^{-1}(s)) = \chi_{A_{(c,d)}}(s)$ with the inclusions $(\varphi(c+), \varphi(d-)) \subset A_{(c,d)} \subset (\varphi(c+), \varphi(d-)]$, hence formula (42) holds. By additivity of the Kurzweil integral, it holds for every step function, and by density, it can be extended to the whole space $G(a, b)$. $\qquad\square$

Lemma 4.9. *Let the hypotheses of Theorem 4.5 hold. Then*

$$\int_0^b f(t) \, \mathrm{d}\varphi(t) = \int_0^{\varphi(b)} f(\varphi^{-1}(s)) \, \mathrm{d}s. \qquad (43)$$

Proof of Lemma 4.9. For $a \in (0, b)$ set

$$\varphi_a(t) = \begin{cases} \varphi(t) & \text{for } t \in (a, b], \\ \varphi(0+) & \text{for } t \in (0, a], \\ 0 & \text{for } t = 0. \end{cases}$$

In other words, we have

$$\varphi_a(t) = \begin{cases} \varphi(0+)(1 - \chi_{\{0\}}(t)) & \text{for } t \in [0, a], \\ \varphi(t) + \chi_{\{a\}}(t)(\varphi(0+) - \varphi(a)) & \text{for } t \in [a, b]. \end{cases}$$

Both integrals $\int_0^a f(t) \, \mathrm{d}\varphi_a(t) = f(0)\varphi(0+)$ and $\int_a^b f(t) \, \mathrm{d}\varphi_a(t)$ exist, hence the integral $\int_0^b f(t) \, \mathrm{d}\varphi_a(t)$ exists and equals the sum of the two, that is,

$$\int_0^b f(t) \, \mathrm{d}\varphi_a(t) = \int_a^b f(t) \, \mathrm{d}\varphi(t) + f(0)\varphi(0+) + f(a)(\varphi(a) - \varphi(0+)). \qquad (44)$$

By Lemma 4.8, we have

$$\int_0^b f(t) \, \mathrm{d}\varphi_a(t) = \int_{\varphi(a)}^{\varphi(b)} f(\varphi^{-1}(s)) \, \mathrm{d}s + f(0)\varphi(0+) + f(a)(\varphi(a) - \varphi(0+)). \qquad (45)$$

The functions φ_a converge uniformly to φ on $[0, b]$. Indeed, for all $t \in [0, b]$, we have $|\varphi_a(t) - \varphi(t)| \le |\varphi(a) - \varphi(0+)| \to 0$. Hence, we can pass to the limit as $a \to 0+$ in (45) and obtain

$$\int_0^b f(t) \, \mathrm{d}\varphi(t) = \lim_{a \to 0+} \int_0^b f(t) \, \mathrm{d}\varphi_a(t) = \lim_{a \to 0+} \int_{\varphi(a)}^{\varphi(b)} f(\varphi^{-1}(s)) \, \mathrm{d}s + f(0)\varphi(0+)$$

$$= \int_{\varphi(0+)}^{\varphi(b)} f(\varphi^{-1}(s)) \, \mathrm{d}s + f(0)\varphi(0+). \qquad (46)$$

Since $\varphi^{-1}(s) = 0$ for $s \in [0, \varphi(0+)]$, it follows that (46) coincides with (43). $\qquad\square$

We are now ready to finish the proof of Theorem 4.5.

Proof of Theorem 4.5. Assume first that ψ is nondecreasing, $\psi(0) = 0$, and $\psi(B) = C$. By Lemmas 4.8 and 4.9, for all $a \in [0, b)$ we have

$$\int_a^b f(t) \, \mathrm{d}(\psi \circ \varphi)(t) = \int_{(\psi \circ \varphi)(a)}^{(\psi \circ \varphi)(b)} f((\psi \circ \varphi)^{-1}(r)) \, \mathrm{d}r. \qquad (47)$$

By Lemma 4.7, $f((\psi \circ \varphi)^{-1}(r)) = (f \circ \varphi^{-1})(\psi^{-1}(r))$ for all $r \in [0, C]$. Using again Lemmas 4.8 and 4.9 with φ replaced by ψ and f replaced by $f \circ \varphi^{-1}$, we obtain

$$\int_{(\psi \circ \varphi)(a)}^{(\psi \circ \varphi)(b)} (f \circ \varphi^{-1})(\psi^{-1}(r)) \, \mathrm{d}r = \int_{\varphi(a)}^{\varphi(b)} (f \circ \varphi^{-1})(s) \, \mathrm{d}\psi(s). \qquad (48)$$

Hence, Theorem 4.5 is proved for nondecreasing ψ satisfying $\psi(0) = 0$. An arbitrary right continuous function with bounded variation ψ can be decomposed into the sum $\psi = \psi_+ - \psi_- + \psi_c$ such that both ψ_+, ψ_- are nondecreasing and right continuous, $\psi_\pm(0) = 0$, and ψ_c is constant. For each of the functions $\psi_\pm, \psi_c$, formula (37) holds, hence it holds for ψ, and the proof is complete. $\square$

5. **Conclusions.** We have proved the substitution formula for the Kurzweil integral where a regulated function is integrated with respect to a function with bounded variation. Using this formula we have shown that a discontinuous generalization of the Prandtl-Ishlinskii operator has the following composition property: $\mathcal{P}_{\psi_1} \circ \mathcal{P}_{\psi_2} = \mathcal{P}_{\psi_1 \circ \psi_2}$ if the primary response curve ψ_2 of the (generalized) Prandtl-Ishlinskii operator $\mathcal{P}_{\psi_2}$ is monotone. The composition formula allowed us to show that a network of interacting (generalized or classical) Prandtl-Ishlinskii operators is effectively equivalent to one (possibly discontinuous) Prandtl-Ishlinskii operator provided that the algebraic system of equations (14) admits a monotone solution. This means, in particular, that if a standard phenomenology of the Prandtl-Ishlinskii model, which assumes that plays do not affect each other, is modified to include relatively weak interactions between the plays, then the resulting network model can be reduced again to an effective Prandtl-Ishlinskii model of non-interacting plays with some changed parameters. This result may explain why the simplified (non-interacting) phenomenology of the Prandtl-Ishlinskii model produces good approximations to real data across multiple applications. However, the model with stronger interactions between the plays can exhibit more complicated hysteresis effects (including non-closed hysteresis loops and ratcheting) than can any (generalized) Prandtl-Ishlinskii operator [16]. Thus, two situations can arise when interactions are introduced into a Prandtl-Ishlinskii system of independent plays: one where interactions result merely in a change of parameters of the plays (and the network model is equivalent to a Prandtl-Ishlinskii operator); and another where complex hysteresis effects such as ratcheting appear. Both cases can be accounted for by just one class of network models. We note that standard models of ratcheting, used for example in the study of fatigue and damage, combine the Prandtl-Ishlinskii model with an additional nonlinearity (see Section 5.4.4 in [9]). However our results suggest that ratcheting can be induced by the strong interaction of elementary constitutive laws, which is ignored in the Prandtl-Ishlinskii formalism. Solvability of the algebraic system (14) within in the class of monotone functions can be interpreted as a criterion that decides which of the two above scenarios will be realized by any particular network of interacting play operators (or, more generally, network of generalized Prandtl-Ishlinskii operators).

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