Project 1 — Math 674

This project must be your own work. No collaboration or use of AI is allowed. It is due by the start of the class on the day of the midterm. No late submissions will be accepted. Please email me the completed project as a single pdf file.

You may use any numerical package or programming language you like. Your submission should include printouts of all programs written, relevant pictures, suitable summaries of simulation results etc.

Question 1 — Stochastic Integrals
In class we performed an explicit calculation of the Itô integral and proved that

$$\int_0^T B_s \, dB_s = \frac{1}{2}B_T^2 - \frac{1}{2}T. \qquad (1)$$

This was evaluated using a 'Riemann-sum' argument where the interval $[0, T]$ was split into subintervals and the function was approximated over the subinterval $[t_i, t_{i+1}]$ by a constant function evaluated at the *left-hand end*.

The purpose of this numerical experiment is to demonstrate that the requirement to use the left-hand value of each subinterval is absolutely essential — unlike the situation for deterministic integrals.

1. Generate a Brownian motion over the interval $[0, 3]$ using a pseudo-random-number generator (such as the `randn` function in MATLAB/OCTAVE). By this I mean generate values of the Brownian motion at a finite number of equally-spaced times in the interval — 2^{15} should be more than sufficient, but if you find that is too computer-intensive by all means reduce the number to some smaller power of 2.
2. Approximate the Itô integral in (**??**) using a Riemann sum over all 2^{15} intervals and thus numerically confirm the result.
3. Now instead use the value of B_s at the mid-point of each sub-interval (instead of the left-hand end) to be the value of the approximating constant function (note that you will need to use 2^{14} subintervals rather than 2^{15} so that the mid-point values are accessible). What value do you get and how does it appear to relate to the quantities B_T, T?
4. Now do the same again but use the right-hand end of each of the 2^{15} subintervals.
5. We will now investigate the rate of convergence of the Riemann sum to the correct answer as the number of subintervals increases. Generate a sequence of approximations for the Itô integral using 2^k subintervals from $k = 5$ to $k = 15$. Plot the magnitude of the errors versus number of subintervals used on a graph with appropriate scales. Comment as much as you can on the rate of convergence.

Question 2 — Numerical Estimation of a PDF

Recall that in class we derived, using the reflection principle, the PDF of the stochastic process $B_t^* = \max_{0\le s\le t} B_s$. By making (many) repeated runs with a suitably small stepsize (say 2^{-8}), produce a numerical plot of the PDF of B_1^* and overlay it with the exact analytic result.