# Math 674 HW solutions Week 1

1.1) Let $T_{m,n}$ = expected time to ^first reach \$n from \$m.

Then $T_{25,18} = T_{25,20} + T_{20,19} + T_{19,18} = 5\,T_{25,24} + T_{20,19} + T_{19,18}$

$T_{25,24} = \frac{1}{3}\left(T_{26,24} + 1\right) + \frac{2}{3}\left(\cancel{T_{24,24}} + 1\right) = \frac{2}{3} + \frac{1}{3}\left(2\,T_{25,24} + 1\right)$

Solving $\Rightarrow$ $T_{25,24} = 3$ $\Rightarrow$ $T_{25,20} = 15$

$T_{20,19} = 0.1\left(1 + \cancel{T_{19,19}}\right) + 0.9\left(1 + T_{21,19}\right) = 0.1 + 0.9\left(1 + T_{21,20} + T_{20,}\right.$

But $T_{21,20} = 3$. Solving $\Rightarrow$ $T_{20,19} = 37$

$T_{19,18} = \frac{1}{3}\left(1 + \cancel{T_{18,18}}\right) + \frac{2}{3}\left(1 + 37 + T_{19,18}\right)$. Solving $\Rightarrow$ $T_{19,18} = 7$

So $T_{25,18} = 15 + 37 + 77 = \underline{129}$

1.2) a) $\binom{2K}{K} 2^{-2K} \sim 2^{-2K} \dfrac{\sqrt{4\pi K}\,(2K)^{2K} e^{-2K}}{\sqrt{2\pi K}\,K^K e^{-K} \sqrt{2\pi K}\,K^K e^{-K}} = \dfrac{1}{\sqrt{\pi K}}$

b) $N = \sum_{n=1}^{\infty} \mathbb{1}(S_n = 0)$

$1^{st}$ way: $E(N) = \sum_{n=1}^{\infty} P(S_n = 0) \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$

$2^{nd}$ way: Let $r = P(S_n = 0 \text{ for some } n \geq 1 \mid S_0 = 0)$

Then $r^2 = P(S_n = 0 \text{ for two } n_1, n_2 \geq 1 \mid S_0 = 0)$ etc.

Summing gives $E(N) = r + r^2 + r^3 + \dots = \frac{r}{1-r}$

But this is $\infty$ $\Rightarrow$ $\underline{\underline{r = 1}}$

c) Save for next week.

1.3) Let $N_K$ = # visits to level $K$ before returning to 0.

Wlog let $K > 0$

$$P(N_K > 0) = P(1^{st}\text{ step} = +1) \times P(\text{hitting } K \text{ before returning to } 0)$$
$$= \frac{1}{2} \times \frac{1}{A+B} \text{ where } A = K-1 \text{ and } B = 1$$
$$= \frac{1}{2K}.$$

$$P(N_K > j+1 \mid N_K > j) = \frac{1}{2} + \frac{1}{2}\frac{K-1}{K} = 1 - \frac{1}{2K}$$

~~prob of getting +1 when at K for (j+1)st time~~

prob of getting $-1$ when at $K$ for $j+1$ th time $\times$ prob of hitting $K$ again before 0 $\left(= \frac{K-1}{A+B} \text{ with } A = \cancel{K-1}, B = K-1\right)$

~~Now~~ Note that $P(N_K = j+1 \mid N_K > j) = \frac{1}{2K}$.

So $P(N_K = 1) = P(N_K > 0)\, P(N_K = 1 \mid N_K > 0) = \frac{1}{2K} \cdot \frac{1}{2K}$.

~~$P(N_K = 2) = P(N_K = 1)\, P(N_K = 2 \mid N_K = 1)$~~

$$P(N_K = 2) = P(N_K > 0)\, P(N_K > 1)\, P(N_K = 2) = \frac{1}{2K}\left(1 - \frac{1}{2K}\right)\cdot\frac{1}{2K}$$
$$P(N_K = m) = - - - - - - = \frac{1}{2K}\left(1 - \frac{1}{2K}\right)^{m-1}\frac{1}{2K}$$

$$\text{So } E(N_K) = \frac{1}{4K^2}\sum_{m=1}^{\infty} m\left(1 - \frac{1}{2K}\right)^{m-1}$$

Now note that $\sum_{m=1}^{\infty} m r^m = (r + r^2 + r^3 + \dots) + (r^2 + r^3 + r^4 + \dots) + \dots$

$$\frac{1}{4K^2}\left(\frac{1}{1-(1-\frac{1}{2K})}\right) = \frac{1}{2K} \qquad \frac{1}{2K}\left(1 - \frac{1}{2K}\right)$$

Sum of all brackets (another geometric series) is $\frac{1}{2K}\frac{1}{1-(1-\frac{1}{2K})} = 1$