Solutions to Chapter 6 Exercises

6.1 [Remark: the means of both Itô Integrals are zero]

a) $\mathrm{var}\left[\int_0^t |B_s|^{\frac{1}{2}} dB_s\right] = E\left[\left(\int_0^t |B_s|^{\frac{1}{2}} dB_s\right)^2\right] \underset{\text{Isometry}}{\overset{\text{Itô}}{=}} E\left[\int_0^t \left(|B_s|^{\frac{1}{2}}\right)^2 ds\right]$

$= E\left[\int_0^t |B_s|\, ds\right] = \int_0^t E[|B_s|]\, ds = \int_0^t E(B_s^*)\, ds$ from Chapter 5

Now $E[B_s^*] = \int_0^\infty \sqrt{\frac{2}{\pi s}}\, e^{-x^2/2s} . x\, dx = \sqrt{\frac{2s}{\pi}}\left[-e^{-x^2/2s}\right]_0^\infty = \sqrt{\frac{2s}{\pi}}$.

So $\int_0^t E(B_s^*)\, ds = \int_0^t \sqrt{\frac{2s}{\pi}}\, ds = \sqrt{\frac{2}{\pi}}\left[\frac{2}{3} s^{3/2}\right]_0^t = \underline{\underline{\frac{2}{3}\sqrt{\frac{2}{\pi}}\, t^{3/2}}}$

b) $\mathrm{var}\left[\int_0^t (B_s+s)^2 dB_s\right] = E\left[\left(\int_0^t (B_s+s)^2 dB_s\right)^2\right] \underset{\text{Isometry}}{\overset{\text{Itô}}{=}} E\left[\int_0^t (B_s+s)^4 ds\right]$

$= E\left[\int_0^t B_s^4 + 4B_s^3 s + 6B_s^2 s^2 + 4B_s s^3 + s^4\, ds\right]$

$= \int_0^t E[B_s^4 + \cancel{4sB_s^3} + 6B_s^2 s^2 + \cancel{4B_s s^3} + s^4\, ds]$ since all odd moments of B_s are 0

$= \int_0^t E[B_s^4] + 6s^2 E[B_s^2] + s^4\, ds$

$= \int_0^t 3s^2 + 6s^3 + s^4\, ds = \underline{\underline{t^3 + \frac{3}{2}t^4 + t^5/5}}$

$$= E\left[\int_0^t B_s^4 + 4B_s^3 s + 6B_s^2 s^2 + 4B_s s^3 + s^4 \, ds\right]$$

$$= \int_0^t E\left[B_s^4 + 4s\cancel{B_s^3} + 6B_s^2 s^2 + \cancel{4B_s s^3} + s^4 \, ds\right]$$ since all odd moments of B_s are 0

$$= \int_0^t E[B_s^4] + 6s^2 E[B_s^2] + s^4 \, ds$$

$$= \int_0^t 3s^2 + 6s^3 + s^4 \, ds = \underline{\underline{t^3 + \tfrac{3}{2} t^4 + t^5/5}}$$

[Note: E and $\int$ can be switched (commute) because E is also really just an integral]

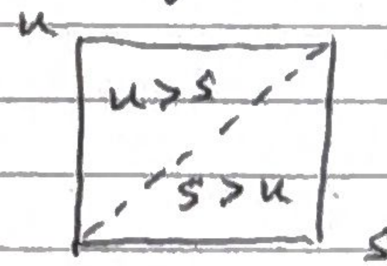
<u>6.2</u>

$$E[I_1] = E\left[\int_0^t B_s \, ds\right] = \int_0^t E[B_s] \, ds = \int_0^t 0 \, ds = 0$$

$$\mathrm{Var}[I_1] = E\left[\left(\int_0^t B_s \, ds\right)^2\right].$$ (Note the Itô Isometry does <u>not</u> apply)

$$= E\left[\left(\int_0^t B_s \, ds\right)\left(\int_0^t B_u \, du\right)\right] = E\left[\iint_0^t B_s B_u \, ds \, du\right] = \iint_0^t E[B_s B_u] \, ds \, du$$

$$= \iint_0^t \min(s,u) \, ds \, du$$. Integral splits into



and evaluates to $2 \times t^3/6 = \underline{\underline{t^3/3}}$

2.

$$E[I_2] = E\left[\int_0^t B_s^2\,ds\right] = \int_0^t E[B_s^2]\,ds = \int_0^t s\,ds = \underline{\tfrac{t^2}{2}}$$

$$Var[I_2] = E\left[\left(\int_0^t B_s^2 ds\right)^2\right] - \left(\tfrac{t^2}{2}\right)^2 = E\left[\left(\int_0^t B_s^2\,ds\right)\left(\int_0^t B_u^2\,du\right)\right] - t^4/4$$

$$= E\left[\iint_0^t B_s^2 B_u^2\,ds\,du\right] - t^4/4 = \iint_0^t E[B_s^2 B_u^2]\,ds\,du - t^4/4$$

$$= 2\iint_{\substack{0\\ s<u}}^{t} E[B_s^2 B_u^2]\,ds\,du - t^4/4$$

$$= 2\iint_{\substack{0\\ s<u}}^{t} E\left[(B_s + (B_u - B_s))^2 B_s^2\right] ds\,du - t^4/4$$

$$= 2\iint_{\substack{0\\ s<u}}^{t} E[B_s^4] + \cancel{2E[B_s^2(B_u - B_s)]} + E[(B_u - B_s)^2 B_s^2]\,ds\,du - t^4/4$$

$$= 2\iint_{\substack{0\\ s<u}}^{t} 3s^2 + 0 + s(u-s)\,du\,ds - t^4/4 \quad \text{by Independent increments}$$

$$= \ldots = \frac{7t^4}{12} - t^4/4 = \underline{\underline{\tfrac{t^4}{3}}}$$

6.3

$$E[U_t] = E\left(\int_0^t f(B_s)\,ds\right) = \int_0^t E[f(B_s)]\,ds$$

$$= 2\iint_{0,\ s<u}^{t} 3s^2 + 0 + s(u-s)\,du\,ds - t^4/4$$ by Independent increments

$$= \dots = \frac{7t^4}{12} - t^4/4 = \underline{\underline{\frac{t^4}{3}}}$$

6.3

$$E[U_t] = E\left(\int_0^t f(B_s)\,ds\right) = \int_0^t E[f(B_s)]\,ds$$

& $E[V_t] = \int_0^t E[f(X_s)]\,ds$. But for each s, B_s and X_s have same distribution so are equal.

Now for the variance.

Let's choose f = Identity for simplicity

Then mean ~~and variance~~ is $= 0$ in both cases.

$$\operatorname{var}[U_t] = E[U_t^2] = E\left[\left(\int_0^t B_s\,ds\right)^2\right] = E\left[\left(\int_0^t B_s\,ds\right)\left(\int_0^t B_u\,du\right)\right] = \frac{t^3}{3} \text{ as before}$$

$$\operatorname{var}\{V_t\} = E[V_t^2] = E\left[\left(\int_0^t X_s\,ds\right)^2\right] = E\left[\left(\int_0^t Z\sqrt{s}\,ds\right)^2\right]$$

$$= E\left[Z^2\left(\int_0^t \sqrt{s}\,ds\right)^2\right] = E\left[Z^2\left(\tfrac{2}{3}t^{3/2}\right)^2\right] = 4/9\, t^3.$$