Solutions to the homework questions for Section 16.8.

1 Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$. Then

$$\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{xy - xy}{(x^2 + y^2)^{\frac{3}{2}}} = 0.$$

2 $\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 1 + 1 = 2.$

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$$\mathbf{F}(x, y, z) = -\frac{GM(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

where G is the universal gravitational constant and M is the mass of the object at the origin causing the gravitational field (note that the force points inwards towards the origin and its magnitude obeys the inverse-square law). Now

$$\begin{aligned}\nabla \cdot \mathbf{F} &= -GM\frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3x^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} - GM\frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3y^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} \\ &\quad -GM\frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3z^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} = 0\end{aligned}$$

except at the origin where the field is undefined.

4 $z = a^2 - r^2$ in cylindrical coordinates so $\mathbf{v} = (a^2 - x^2 - y^2)\mathbf{k}$ and $\nabla \cdot \mathbf{F} = 0$.

5 $\frac{\partial}{\partial x}(y - x) = -1$, $\frac{\partial}{\partial y}(z - y) = -1$ and $\frac{\partial}{\partial z}(y - x) = 0$. Therefore $\nabla \cdot \mathbf{F} = -2$ and the volume integral of $\nabla \cdot \mathbf{F}$ is just -2 times the volume of the box which is -16.

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a) This was one of the questions on the fourth test. The divergence of the curl is always 0.

b) By the Divergence Theorem, the outward flux of any vector field across any closed surface is 0 — **if** that vector field is the curl of some other vector field.

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a) The volume integral on the LHS represents the total mass inside the region at time t. The double integral on the RHS represents the mass flux across the boundary into the region (note the - sign). Thus the equation states that the rate of change of total mass (with respect to time) equals the rate of mass transport into the region (and so no mass is being created or destroyed inside the region).

b) By Leibnitz's Rule (which says that the time derivative can be moved inside the space integral) the LHS of a) becomes $\int\int\int_D \frac{\partial p}{\partial t}\, dV$. By the Divergence Theorem the RHS of a) can be replaced by $\int\int\int_D -\nabla \cdot (p\mathbf{v})\, dV$. Since these two triple integrals must be equal for an arbitrary volume D the integrands must be equal and this gives the result.