MATH 203 Linear Algebra
Solutions to Review Problems for Test I

1. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$. Compute the indicated vector.

   (a) $\mathbf{u} + 2\mathbf{v} - 4\mathbf{w} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} - 4\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+8+12 \\ 3+4 \\ -2-2-8 \end{bmatrix} = \begin{bmatrix} 19 \\ 7 \\ -12 \end{bmatrix}$

   (b) $4\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = 4\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} + 4\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4-8-12 \\ 12-4 \\ -8+2+8 \end{bmatrix} = \begin{bmatrix} -24 \\ 8 \\ 2 \end{bmatrix}$

2. Find all scalars c, if any exist, such that the given statement is true.

   (a) The vector $\begin{bmatrix} c \\ -c \\ 4 \end{bmatrix}$ is a scalar multiple of the vector $\begin{bmatrix} -2 \\ 2 \\ 20 \end{bmatrix}$.

   The vector $\begin{bmatrix} c \\ -c \\ 4 \end{bmatrix}$ is a scalar multiple of the vector $\begin{bmatrix} -2 \\ 2 \\ 20 \end{bmatrix}$ if and if there is a scalar $s \in \mathbb{R}$ such that

$$\begin{bmatrix} c \\ -c \\ 4 \end{bmatrix} = s\begin{bmatrix} -2 \\ 2 \\ 20 \end{bmatrix} \Rightarrow 4 = 20s \Rightarrow s = \frac{4}{20} = \frac{1}{5} \Rightarrow c = -2s = -2\frac{1}{5} = -\frac{2}{5}$$

   (b) The vector $\begin{bmatrix} 3 \\ -2 \\ c \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

   The vector $\begin{bmatrix} 3 \\ -2 \\ c \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ if and only if there exists scalars $s_1, s_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 3 \\ -2 \\ c \end{bmatrix} = s_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + s_2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Leftrightarrow \begin{array}{rcrcr} s_1 & + & s_2 & = & 3 \\ 2s_1 & & & = & -2 \\ -s_1 & + & 3s_2 & = & c \end{array}$$

   Solving the linear system

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & -2 \\ -1 & 3 & c \end{bmatrix} \xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -8 \\ 0 & 4 & 3+c \end{bmatrix} \xrightarrow[-4R_2+R_3]{-\frac{1}{2}R_2} \begin{bmatrix} \textcircled{1} & 1 & 3 \\ 0 & \textcircled{1} & 4 \\ 0 & 0 & -13+c \end{bmatrix}$$

   For the system to be consistent $-13 + c = 0 \Leftrightarrow c = 13$.

3. Describe all solutions of a linear system whose corresponding augmented matrix can be row reduced to the given matrix. Write the solution, if it exists, in vector parametric form.

   (a)

$$\left[\begin{array}{ccc|r} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{-2R_2+R_1} \left[\begin{array}{ccc|r} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-3R_3+R_1]{-2R_3+R_2} \left[\begin{array}{ccc|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}$$

(b) Basic variables: x_1, x_3, x_5, free variables: x_2, x_4

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & 0 & -4 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{ccccccccc} x_1 & & x_2 & & & + & 3x_4 & & & = & -4 \\ & & & & x_3 & - & x_4 & & & = & 0 \\ & & & & & & & & x_5 & = & -2 \end{array}$$

$$\Rightarrow \begin{array}{rcl} x_1 & = & -4 - x_2 - 3x_4 \\ x_3 & = & x_4 \\ x_5 & = & -2 \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4 - x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

4. Find all solutions of the given linear system. If the solution exists, write it in vector parametric form.

(a)

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & 2x_3 & = & 4 \\ 2x_1 & + & 7x_2 & - & x_3 & = & -2 \\ 2x_1 & + & 9x_2 & - & 7x_3 & = & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 4 \\ 2 & 7 & -1 & -2 \\ 2 & 9 & -7 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 4 \\ 0 & -1 & 3 & -10 \\ 0 & 1 & -3 & -7 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} \textcircled{1} & 4 & -2 & 4 \\ 0 & \textcircled{-1} & 3 & -10 \\ 0 & 0 & 0 & \textcircled{-17} \end{array}\right]$$

Since there is a pivot in the last column, the system is inconsistent.

(b)

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 2x_3 & + & x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & - & 3x_4 & = & 0 \\ 9x_1 & - & 3x_2 & - & x_3 & - & 7x_4 & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 2 & -1 & 1 & -3 & 0 \\ 9 & -3 & -1 & -7 & 4 \end{bmatrix} \xrightarrow[-9R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & -1 & 5 & -5 & -12 \\ 0 & -3 & 17 & -16 & -50 \end{bmatrix} \xrightarrow[3R_2+R_3]{-R_2} \begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & 1 & -5 & 5 & -12 \\ 0 & 0 & 2 & -1 & -14 \end{bmatrix}$$

$$\xrightarrow[5R_3+R_2]{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & 1 & 0 & -\frac{5}{2} & 23 \\ 0 & 0 & 1 & -\frac{1}{2} & -7 \end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & -8 \\ 0 & \textcircled{1} & 0 & \frac{5}{2} & -23 \\ 0 & 0 & \textcircled{1} & -\frac{1}{2} & -7 \end{bmatrix}$$

Basic variables: x_1, x_2, x_3, free variable(s): x_4

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 \\ -23 - \frac{5}{2}x_4 \\ -7 + \frac{1}{2}x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 \\ -23 \\ -7 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -\frac{5}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

5. Determine whether the columns of the given matrix span $\mathbb{R}^4$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[3R_1+R_4]{-R_1+R_3} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 3 & -4 \end{bmatrix} \xrightarrow{\frac{3}{2}R_3+R_4} \begin{bmatrix} \textcircled{1} & 0 & 1 & -1 \\ 0 & \textcircled{-1} & -3 & 4 \\ 0 & 0 & \textcircled{-2} & 3 \\ 0 & 0 & 0 & \textcircled{1/2} \end{bmatrix}$$

By Theorem 4, since there is a pivot position in every row, the columns of the matrix span $\mathbb{R}^4$.

6. Determine whether the given set of vectors is dependent or independent.

(a) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^3$. Solving $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$.

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \\ 1 & 3 \end{bmatrix} \xrightarrow[R_1+R_3]{2R_1+R_2} \begin{bmatrix} -1 & 2 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \textcircled{-1} & 2 \\ 0 & \textcircled{5} \\ 0 & 0 \end{bmatrix}$$

Since there is no free variable, the vectors are linearly independent.

(b) $\left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 13 \\ 4 \\ 7 \end{bmatrix} \right\}$ in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 5 & 13 \\ -1 & 6 & 4 \\ 3 & 2 & 7 \end{bmatrix} \xrightarrow[-3R_1+R_4]{-4R_1+R_2, R_1+R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 9 & 9 \\ 0 & 5 & 5 \\ 0 & 5 & 4 \end{bmatrix} \xrightarrow[-5R_2+R_4]{\frac{1}{9}R_2, -5R_2+R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} \textcircled{1} & -1 & 1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{-1} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is no free variable, the vectors are linearly independent.

7. Find all scalars s, if any exist, such that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ s \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -s \\ 0 \end{bmatrix}$ are linearly independent.

Suppose that $x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ s \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -s \\ 0 \end{bmatrix} = \mathbf{0}$ We solve the homogeneous linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & sx_2 & - & sx_3 & = & 0 \\ x_1 & + & 3x_2 & & & = & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & s & -s & 0 \\ 1 & 3 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{-R_1+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & s & -s & 0 \end{array}\right] \xrightarrow{-sR_2+R_3} \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 0 \\ 0 & \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the homogeneous system has a non-trivial solution (there is a free variable), the vectors are dependent for all values of $s \in \mathbb{R}$.

8. Show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3, T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2, x_1 + 3x_2)$

Write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors. Since $\mathbf{x}$ has 2 entries, A has 2 columns. Since $T(\mathbf{x})$ has 3 entries, A has 3 rows.

$$\begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x_1, x_2, x_3) = (x_1 - x_2 + 3x_3, x_1 + x_2 + x_3, x_1)$

Write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors. Since $\mathbf{x}$ has 3 entries, A has 3 columns. Since $T(\mathbf{x})$ has 3 entries, A has 3 rows.

$$\begin{bmatrix} x_1 - x_2 + 3x_3 \\ x_1 + x_2 + x_3 \\ x_1 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

9. (a) If $T(1,0)=(3,-1)$ and $T(0,1)=(-2,5)$, Find $T(4,-6)$.

$$\begin{aligned}(4,-6)=4(1,0)+(-6)(0,1)\Rightarrow T(4,-6)&=T(4(1,0)+(-6)(0,1))=T(4(1,0))+T((-6)(0,1))\\&=4T(1,0)+(-6)T(0,1)=4(3,-1)-6(-2,5)\\&=(12,-4)-(12,-30)=(24,-34).\end{aligned}$$

(b) If $T(-1,2)=(1,0,0)$ and $T(2,1)=(0,1,2)$, Find $T(0,10)$.

Let $\mathbf{w}=\begin{bmatrix}0\\10\end{bmatrix}, bvv_1\begin{bmatrix}-1\\2\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}2\\1\end{bmatrix}$. We first express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is $\mathbf{w}=x_1\mathbf{v}_1+x_2\mathbf{v}_2$. Row reducing the augmented matrix:

$$\begin{bmatrix}-1&2&0\\2&1&10\end{bmatrix}\xrightarrow[-2R-1+R_2]{-1R_1}\begin{bmatrix}1&-2&0\\0&5&10\end{bmatrix}\xrightarrow[2R_2+R_1]{\frac{1}{5}R_2}\begin{bmatrix}1&0&4\\0&1&2\end{bmatrix}$$

So $\mathbf{w}=4\mathbf{v}_1+2\mathbf{v}_2$ then

$$\begin{aligned}T(0,10)&=T(4(-1,2)+2(2,1))=4T(-1,2)+2T(2,1)\\&=4(1,0,0)+2(0,1,2)=(4,0,0)+(0,2,4)=(4,2,4)\end{aligned}$$

10. Give the matrix representation of the rotation of the plane counterclockwise about the origin through an angle of

(a) $30° \Rightarrow \begin{bmatrix}\sqrt{3}/2&-1/\sqrt{2}\\1/2&\sqrt{3}/2\end{bmatrix}$

(b) $90° \Rightarrow \begin{bmatrix}0&-1\\1&0\end{bmatrix}$

(c) $135° \Rightarrow \begin{bmatrix}-1/\sqrt{2}&-1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}$

11. Show that the linear transformation

$$T(x,y)=\begin{bmatrix}1&0\\0&r\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\ry\end{bmatrix}$$

We see that the first component is left unchanged, while the second (vertical) component is multiplied by r resulting in

(a) A vertical expansion, if $r>1$

(b) If $r<-1$, we have a vertical expansion by a factor $|r$ = -r, followed by a reflection about the x-axis.