1a. $-4x + ry = 2$ — e1

$12x + 5y = -4$ — e2

Multiplying e1 by 3, we have

$$
\begin{array}{r}
-12x + 3ry = 6 \\
+ \quad 12x + 5y = -4 \\
\hline
0 + (3r+5)y = 2
\end{array}
$$

(i) The solution is empty when

$3r + 5 = 0$, that is

$3r = -5$

(ii) Contains unique solution

when $3r + 5 \neq 0$

(iii) The solution set does not contain many solutions since there is no free variable.

1b. $\begin{bmatrix} 1 & r & 4 \\ 3 & 6 & 8 \end{bmatrix} \begin{matrix} - R1 \\ - R2 \end{matrix}$

Applying row operation,

$R2 \to -3R1 + R2$, we have

$$\begin{bmatrix} 1 & r & 4 \\ 0 & -3r+6 & -4 \end{bmatrix}$$

The augmented matrix represent a consistent linear system when $-3r + 6$ is not zero.

That is, it is inconsistent when $-3r + 6 = 0$

$$\Rightarrow r = 2$$

$\therefore$ The augmented matrix represent a consistent linear system for all values of r not equal to 2.

2a.

$$\begin{array}{c} \left[\begin{array}{cccc} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{array}\right] \end{array} \begin{array}{l} — R1 \\ — R2 \\ — R3 \end{array}$$

We will first transform the augmented matrix into rows reduced echelon form.

$R2 \longrightarrow 3R1 + R2, \quad R3 \longrightarrow 2R1 + R3$

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Obtaining $0=0$ in R2 and R3 implies x_2 and x_3 are free variables.

We now make the pivot entry of the pivot column one. That is

$$\begin{bmatrix} 1 & -4/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_2 - 2/3\, x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

2b.

(i) A system of equations equivalent to the vector equation

$$x\begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + y\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$ is

$$6x - 3y = 1$$

$$-x + 4y = -7$$

$$5x \qquad = -5$$

(ii) A system of equations equivalent to the vector equation

$$x\begin{bmatrix}-2\\3\end{bmatrix} + y\begin{bmatrix}8\\5\end{bmatrix} + z\begin{bmatrix}1\\-6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

is

$$-2x + 8y + z = 0$$

$$3x + 5y - 6z = 0.$$

3a.

(i) The product is defined because the number of columns of A is equal to the number of entries of x.

So,

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4\times 3 + 2\times 1 \\ 1\times 3 + 6\times 1 \\ 0\times 3 + 1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12+2 \\ 3+6 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \\ 1 \end{bmatrix}$$

(ii) The product is undefined because the number of columns of A is not equal to the entries of x.

36. The augumented matrix is

pivot

$$\begin{array}{c l} \begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} & \begin{array}{l} - R1 \\ - R2 \\ - R3 \end{array} \end{array}$$

Applying row operations, we have

$R1 \longleftrightarrow R1, \quad R2 \longrightarrow 2R1 + R2, \quad R3 \longrightarrow -2R1 + R3$

pivot

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -17 & 17 & 0 \\ 0 & 12 & 9 & 0 \end{bmatrix}$$

Next, $R1 \longleftrightarrow R1, \quad R2 \longleftrightarrow -\frac{1}{17} R2, \quad R3 \longleftrightarrow R3$

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 12 & 9 & 0 \end{bmatrix}$$

Next, $R \longleftrightarrow R1$, $R2 \longleftrightarrow R2$,

$R3 \longrightarrow -12R2 + R3$

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 21 & 0 \end{bmatrix}$$

From the last row, we have

$$21 = 0.$$

$\therefore$ The system has no nontrivial solution. That is, the system has only the trivial solution.

4a. The vectors

(i) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

are linearly dependent because there are more vectors than the number of entries ($p > n$).

(ii) The vectors

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are linearly dependent because the set contains the zero vector.

4b.

$$u = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \quad v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and}$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$$

$$T(u) = Au = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + 0\times 0 + -2\times -4 \\ -2\times 1 + 1\times 0 + 6\times -4 \\ 3\times 1 + -2\times 0 + -5\times -4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -26 \\ 23 \end{bmatrix}$$

$$T(v) = Av = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times a + 0 \times b + -2 \times c \\ -2 \times a + 1 \times b + 6 \times c \\ 3 \times a + -2 \times b + -5 \times c \end{bmatrix}$$

$$= \begin{bmatrix} a - 2c \\ -2a + b + 6c \\ 3a - 2b - 5c \end{bmatrix}$$

5. Given that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with

$$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2) \text{ --- } (*)$$

We want to find x such that

$$T(x) = (-1, 4, 9) \text{ --- } (*)$$

From $(*)$ and $(**)$, we have

$$(x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2) = (-1, 4, 9)$$

$$\Rightarrow \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 4 \\ 3x_1 - 2x_2 &= 9 \end{aligned}$$

The angumented matrix is

$$\begin{array}{cl} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} & \begin{array}{l} \text{— R1} \\ \text{— R2} \\ \text{— R3} \end{array} \end{array}$$

Applying row operation, we have

$R1 \leftrightarrow R1$, $R2 \rightarrow R1 + R2$, $R3 \rightarrow -3R1 + R3$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix}$$

Next, $R \leftrightarrow R1$, $R2 \leftrightarrow R2$, $R3 \rightarrow -4R2 + R3$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Next, $R1 \rightarrow 2R2 + R1$, $R2 \leftrightarrow R2$
$R3 \leftrightarrow R3$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 &= 5 \\ x_2 &= 3 \end{aligned} \qquad \therefore x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$