MIDTERM 1 PRACTICE PROBLEMS–SOLUTIONS

1. Problem 1

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 - x_2 + 3x_3 &= -5 \\ -x_2 + x_3 &= 0 \end{aligned}$$

(a) Write the augmented matrix.
(b) Write the system in vector form.
(c) Write the system in matrix form $A\vec{x} = \vec{b}$ (tell us what $A, \vec{x}$, and $\vec{b}$ are for the system above).
(d) Is the system consistent? If yes, is the solution unique?

Solution: (a) Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & -5 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

(b) Vector form:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

where

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}$$

(c) Matrix form:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & -1 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}$$

(d) The RREF of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the last row is all zeros except for a $1 \neq 0$ in the right-most entry, the system is not consistent.

2. Problem 2

Let a, b, c, d be real numbers. Assume $a \neq 0$ and $ad - bc \neq 0$. Find the reduced row echelon form of A where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Record the row operations you used and point out where you need to use the assumptions $a \neq 0$ and $ad - bc \neq 0$.

Solution: Since $a \neq 0$ we can divide R1 by a (multiply by $1/a$):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix}$$

Subtract c times R1 from R2:

$$\begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a \\ 0 & d - bc/a \end{bmatrix}$$

Since $ad - bc \neq 0$ and $a \neq 0$ we know $(1/a)(ad - bc) = d - bc/a \neq 0$, so we can divide R2 by it:

$$\begin{bmatrix} 1 & b/a \\ 0 & d - bc/a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

Last we subtract b/a times R2 from R1:

$$\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Problem 3

Mark each statement as True or False.

(a) Consider a system of equations $A\vec{x} = \vec{b}$. Let B be the reduced row echelon form (RREF) of A. If every row of B contains a pivot (or leading 1), then the system is guaranteed to have a unique solution.

(b) Let A be a matrix and $\vec{v}$ a vector. The product $A\vec{v}$ is well-defined if and only if the number of rows of A equals the number of entries of $\vec{v}$.

(c) Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ be a matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Suppose $\vec{a}_3 = -2\vec{a}_1 + 3\vec{a}_2$. Then the equation $A\vec{x} = \vec{0}$ has infinitely many solutions.

(d) Let $\vec{a}_1, \vec{a}_2$ be non-zero vectors in $\mathbb{R}^2$. Then $\{\vec{a}_1, \vec{a}_2\}$ is guaranteed to be linearly independent.

(e) If $\vec{v}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$, then

$$Span\{\vec{u}_1, \vec{u}_2, \vec{v}\} = Span\{\vec{u}_1, \vec{u}_2\}$$

(f) The function

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y - 1 \\ x \end{bmatrix}$$

is a linear transformation.

(g) Let A be the matrix of a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$. Suppose that the columns of A span $\mathbb{R}^3$. Then T is onto.

Solution:

(a) False (there could be columns of B that do not contain a pivot, and if this happens there will be free variables in the general solution, and the solution will not be unique)
(b) False (it's the number of *columns* of A that must equal the number of entries of $\vec{v}$)
(c) True (if the columns of A are linearly dependent, then $A\vec{x} = \vec{0}$ has a non-trivial solution, and therefore has infinitely many solutions)
(d) False (the vectors could be parallel)
(e) True (any linear combination of $\vec{u}_1, \vec{u}_2, \vec{v}$ could be turned into a linear combination of just $\vec{u}_1, \vec{u}_2$ by replacing $\vec{v}$ by its formula in terms of $\vec{u}_1, \vec{u}_2$)
(f) False (the -1 in the first component makes this not a linear transformation)
(g) True (the range of T is the span of the columns of A, so the range is $\mathbb{R}^3$, the whole codomain, and therefore T is onto)

4. Problem 4

Let $\vec{b}$ be a vector in $\mathbb{R}^3$ with components b_1, b_2, b_3. Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= b_1 \\ 2x_1 - x_2 + 3x_3 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

(a) In order for the system to be consistent, the numbers b_1, b_2, b_3 must obey a linear equation. Determine this equation.
(b) Describe the set S of all $\vec{b}$ such that the system is consistent by expressing S as the span of a collection of vectors in $\mathbb{R}^3$.
(c) Assuming $\vec{b}$ belongs to S, how many free variables appear in the general solution of the original system of equations?

Solution: (a) Row-reducing the augmented matrix leads to

$$\begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & -1 & (2b_1 - b_2)/5 \\ 0 & 0 & 0 & (2/5)b_1 - (1/5)b_2 + b_3 \end{bmatrix}$$

and so the equation is

$$(2/5)b_1 - (1/5)b_2 + b_3 = 0$$

or if we multiply through by 5 to clear the denominator

$$2b_1 - b_2 + 5b_3 = 0$$

(b) There are two methods. Method 1: Find the general solution of the equation in (a). The variables b_2, b_3 are free and $b_1 = (1/2)b_2 - (5/2)b_3$. So the general solution in vector form is

$$\vec{b} = \begin{bmatrix} (1/2)b_2 - (5/2)b_3 \\ b_2 \\ b_3 \end{bmatrix} = b_2 \begin{bmatrix} (1/2) \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -(5/2) \\ 0 \\ 1 \end{bmatrix}$$

and so

$$S = Span\{\vec{v}_1, \vec{v}_2\}, \quad \vec{v}_1 = \begin{bmatrix} (1/2) \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -(5/2) \\ 0 \\ 1 \end{bmatrix}$$

Method 2: Write the original system in vector form, and observe that it says $\vec{b}$ is a linear combination of 3 vectors, and therefore

$$S = Span\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}, \quad \vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

(c) 1 free variable (the column of the RREF corresponding to x_3 does not contain a pivot, so x_3 is free).

5. Problem 5

Let A be a matrix with columns $\vec{a}_1, ..., \vec{a}_k \in \mathbb{R}^n$. Let $\vec{v}$ be a vector with entries $v_1, ..., v_\ell$.

(a) What condition must k, ℓ satisfy in order for the product $A\vec{v}$ to be defined?

(b) Give the formula for the product $A\vec{v}$ (assuming the condition in (a) holds).

(c) Calculate the product $A\vec{v}$ where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Solution: (a) $k = \ell$

(b) Assuming $k = \ell$,

$$A\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_k\vec{a}_k$$

(c)

$$A\vec{v} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$