PROBLEM ONE: True or False

Mark each statement True or False. Justify each answer or give a counter example.

- a) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent

False . Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

- b) The columns of a 4 x 5 matrix are linearly dependent.

True . More vectors than the number of entries in each vector

- c) $(AB)^T = A^T B^T$.

False . $(AB)^T = B^T A^T$

- d) If A is a 2 x 6 matrix and T is a linear transformation defined by $T(\vec{x}) = A\vec{x}$, then the domain of T is $\mathbb{R}^2$.

False . Domain $(T) = \mathbb{R}^6$

- e) If A and B are square matrices of the same size then $AB = BA$.

False , in general $AB \neq BA$.

PROBELM TWO: Linear independence

- a)Use a homogeneous system to determine if the vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}$ are linearly independent.

$$\vec{a_1} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

.

Put the 3 vectors as columns of a matrix A then check if the homogeneous system $A\vec{x} = \vec{0}$ has the trivial solution only.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right] \xrightarrow{R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

No free variables
$\Longrightarrow$ Trivial solution only
$\Longrightarrow \{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ is a linearly independent set

- b) Explain how the solution(s) of your homogeneous system determines whether the set above is linearly independent or linearly dependent.

Homogeneous system

Trivial solution only	Non trivial solution
↓	↓
No free variables	at least one free variable
↓	↓
columns of A are L.I	columns of A are L.D

PROBLEM THREE: Matrix of a linear transformation

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - x_2 \\ 2x_2 \end{bmatrix}$

- a) Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$ and let c be any scalar. Prove that T is a linear transformation.

1) $T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} 4(u_1+v_1) - (u_2+v_2) \\ 2(u_2+v_2) \end{bmatrix}$

$= \begin{bmatrix} (4u_1 - u_2) + (4v_1 - v_2) \\ 2u_2 + 2v_2 \end{bmatrix} = \begin{bmatrix} 4u_1 - u_2 \\ 2u_2 \end{bmatrix} + \begin{bmatrix} 4v_1 - v_2 \\ 2v_2 \end{bmatrix}$

$= T(\vec{u}) + T(\vec{v})$ ✓

2) $T(c\vec{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 4(cu_1) - cu_2 \\ 2(cu_2) \end{bmatrix} = \begin{bmatrix} c(4u_1 - u_2) \\ c(2u_2) \end{bmatrix}$

$= c\begin{bmatrix} 4u_1 - u_2 \\ 2u_2 \end{bmatrix} = cT(\vec{u})$ ✓

$\Rightarrow$ T is a linear transformation

- b) Find the standard matrix A of T.

$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{bmatrix} = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$

$= \begin{bmatrix} 4(1) - 0 & 4(0) - 1 \\ 2(0) & 2(1) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$

- c) Is T one-to-one? Prove your answer using the matrix A.

The 2 columns of A are linearly independent
(2 vectors that are not multiple of each other)

$\Rightarrow$ T is one-to-one.

- d) Is T onto? Prove your answer using the matrix A.

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

A has 2 pivot elements $\Rightarrow$ 2 pivot columns

$\Rightarrow$ The columns of A span $\mathbb{R}^2$

$\Rightarrow$ T is onto.

PROBLEM FOUR: Matrix Operations
Find $(AB)^T$ given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Can use 1 of 2 methods:

same answer

1) Calculate AB then take the transpose $(AB)^T$

or 2) find A^T and B^T then calculate $(AB)^T$ by $(AB)^T = B^T A^T$

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & 0 \\ -4 & -8 & -12 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 1 & 1 & -4 \\ 1 & 0 & -8 \\ -3 & 0 & -12 \end{bmatrix}$$

PROBLEM FIVE: Solution of a system of linear equations
Find all solutions, if any exist, of the following system. If the system has multiple solutions, write the solution set as the Span of some vectors.

$$\begin{array}{rrrl} x_1 & +3x_2 & -5x_3 & = 0 \\ x_1 & +4x_2 & -8x_3 & = 0 \\ -3x_1 & -7x_2 & +9x_3 & = 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} x_1 \quad + 4x_3 = 0 \\ x_2 - 3x_3 = 0 \end{array} \implies \begin{array}{l} x_1 = -4x_3 \\ x_2 = 3x_3 \end{array}, \quad x_3 \text{ free variable}$$

$$\implies \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\text{Solution} = \text{Span}\left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \right\}$$