Math 203 EXAM 1 Review

Sections covered: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.9

LEARNING OUTCOMES:

- 1- Recognize a system of linear equations
- 2- Reduce a system into echelon form using row operations
- 3- Interpret the solution(s) of a system of linear equations
- 4- Solve a system using backward substitution
- 5- Visualize a matrix-vector product $A\vec{x} = \vec{b}$ as a linear combination of the columns of A, with weights being the entries in $\vec{x}$.
- 6- Understand the relation between the consistency of a system and linear combinations of the columns.
- 7- Compute a matrix-vector product.
- 8- Determine if a homegenous system has a nontrivial solution
- 9- Determine if a set is linearly independent using the solution(s) of a homogeneous system
- 10- Determine if some sets are linearly independent using shortcut rules
- 11- Determine if a homegenous system has a nontrivial solution
- 12- Determine if a set is linearly independent using the solution(s) of a homogeneous system
- 13- Determine if some sets are linearly independent using shortcut rules

PROBLEM ONE: Solution of a system of linear equations

Find all solutions, if any exist, of the following system. If the system has multiple solutions, write the solution set as the Span of some vectors.

$$\begin{array}{cccccc}
x_1 & +x_2 & -2x_3 & -x_4 & +6x_5 & =0 \\
2x_1 & +2x_2 & -4x_3 & +2x_4 & +14x_5 & =0 \\
x_1 & +x_2 & -2x_3 & +5x_4 & +9x_5 & =0 \\
-3x_1 & -3x_2 & +6x_3 & -15x_4 & -27x_5 & =0
\end{array}$$

PROBLEM TWO: True or False

Mark each statement True or False. Justify each answer or give a counter example.

- a) A set of 2 vectors in $\mathbb{R}^4$ is linearly independent.
- b) If the transformation T is onto and A is its standard matrix, then the equation $A_{nxn}\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^n$.
- c)If A is a 2 x 6 matrix and T is a linear transformation defined by $T(\vec{x}) = A\vec{x}$, then the domain of T is $\mathbb{R}^2$.
- d) If T is one-to-one and A is its standard matrix, then the equation $A\vec{x} = \vec{b}$ has a free variable.
- e) If $\vec{b}$ is a linear combination of the columns of the matrix A then the system $A\vec{x} = \vec{0}$ has only the trivial solution.
- f) The mapping T: $\vec{x} \to cos(\vec{x})$ is one-to-one for $\vec{x} \in \mathbb{R}$.
- g) If A is a 5x3 matrix with 2 pivot positions then the equation $A\vec{x} = \vec{0}$ has only the trivial solution.
- h) The columns of a 3x4 matrix are linearly dependent.
- i) A homogeneous system is always consistent.
- j) If A is a 3x2 matrix then the transformation $\vec{x} \to A\vec{x}$ must be onto.

PROBELM THREE: Linear combinations and Independence

Determine if $\vec{b}$ is a linear combination of $\vec{a_1}, \vec{a_2}, \vec{a_3}$; find the weights if possible.

$$\vec{a_1} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 8 \\ 16 \\ -2 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}$$

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PROBLEM FOUR: Matrix of a linear transformation

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\vec{e_1}) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$, $T(\vec{e_2}) = T\left(\begin{bmatrix} 6 \\ 1 \end{bmatrix}\right)$, and $T(\vec{e_3}) = T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$

- a) Find $T(\vec{x})$ for any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$.

- b) Find the standard matrix A of T.
- c) Is T one-to-one? Prove your answer using the matrix A.
- d) Is T onto? Prove your answer using the matrix A.
- e) Explain why $A\vec{x} = \vec{0}$ has a non trivial solution.

PROBLEM FIVE:

Use the row reduction process to solve the following linear system. If no solution, briefly justify why.

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_3 &= 4 \end{aligned}$$

PROBLEM SIX:

Consider a linear system whose **augmented** matrix is given by

$$\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix}$$

Find a value of h that makes the associated linear system inconsistent. There is more than one value of h that works.

PROBLEM SEVEN:

Let $a_1 = \begin{bmatrix} [r]1 \\ -2 \\ 0 \end{bmatrix}$, $a_2 \begin{bmatrix} [r]0 \\ 1 \\ 2 \end{bmatrix}$, $a_3 = \begin{bmatrix} [r]5 \\ -6 \\ 8 \end{bmatrix}$, $b = \begin{bmatrix} [r]2 \\ -1 \\ 6 \end{bmatrix}$. Is b a linear combination of a_1, a_2, and a_3? Is b in $Span\{a_1,\ a_2, a_3\}$?

PROBLEM EIGHT:

Find a system of linear equations that is equivalent to (has the same solution set as) the given vector equation below.

$$x_1 \begin{bmatrix} [r]6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} [r]-3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} [r]1 \\ -7 \\ -5 \end{bmatrix}$$

PROBLEM NINE:

True or False: If v_1, v_2, v_3 and v_4 are vectors in $\mathbb{R}^4$ and $v_3 = 2v_1 + v_2$ then v_1, v_2, v_3 and v_4 are linearly dependent. Briefly justify.

PROBLEM TEN: True/False: Justify or give a counterexample.

- (Chapter 1) If none of the vectors in the set $S = \{v_1, v_2, v_3\}$ in $\mathbb{R}^3$ is a multiple of one of the other vectors, then S is linearly independent.
- (Chapter 1) If A is a 6×5 matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^5$ onto $\mathbb{R}^6$
- (Chapter 2) If A and B are $m \times n$ matrices, then both AB^T and A^TB are defined.
- (Chapter 2) If A and B are square and invertible, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.
- (Chapter 1) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle θ, then T is a linear transformation.

PROBLEM ELEVEN: Chapter 1

Determine h and k for

$$\begin{aligned} -2x_1 + hx_2 &= 1 \\ 6x_1 + kx_2 &= -2 \end{aligned}$$

such that the solution set of the system

- is empty
- contains a unique solution
- contains infinitely many solutions.

PROBLEM TWELVE (Chapter 1):

Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$. Show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is also linearly independent.

PROBLEM THIRTEEN: (Chapter 2)

If A, B and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution X? If so, find it.

PROBLEM FOURTEEN (Chapter 2):

Compute the following if it is defined. If an expression is undefined, explain why, given the matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

- $C - 2A$
- $A + B$
- AB
- BC
- A^{-1}

PROBLEM FIFTEEN (Chapter 2):

Given T as a linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - 9x_2 \\ -5x_1 + 8x_2 \end{bmatrix}$$

from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

PROBLEM SIXTEEN:

Write the general solution in parametric vector form to the following linear system:

$$\begin{cases} x_1 - x_2 + 3x_3 = 1 \\ 3x_2 + 3x_3 = 6 \end{cases}$$

PROBLEM SEVENTEEN:

Let $S = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ and let $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

1. Is $\mathbf{b}$ in $Span(S)$?
2. Write $\mathbf{b}$ as a linear combination of the vectors in S, if possible.
3. Is the set S linearly independent?

PROBLEM EIGHTEEN:

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & h \end{bmatrix}$.

1. For which h do the columns of A span $\mathbb{R}^2$?
2. For which h are the columns of A linearly independent?
3. For which h is the range of the transformation $T(\mathbf{x}) = A\mathbf{x}$ equal to the codomain?
4. For which h is the system $A\mathbf{x} = \mathbf{0}$ consistent?

PROBLEM NINETEEN:

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

1. Find C^{-1}, if possible.
2. Find AB, if possible.
3. Find BA, if possible.

PROBLEM TWENTY:

Give the solution set for the system using elementary row operations to put in reduced row echelon form, and then reading off the solution.

$$\begin{aligned} 2x + 6z &= 10 \\ -x + 8y + 5z &= 3 \\ x - 2y + z &= 3 \end{aligned}$$

PROBLEM TWENTY ONE:

For the system above, write the associated homogeneous system and give its solution set.

PROBLEM TWENTY TWO:

Write the general solution in parametric form of the system of equations whose **augmented matrix** is given as

$$\begin{bmatrix} 1 & 2 & -5 & 0 & -5 \\ 0 & 1 & -6 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

PROBLEM TWENTY THREE:

Give a geometric description of Span $\{v_1, v_2\}$ for the vectors

$$v_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 12 \\ 3 \\ 9 \end{bmatrix}$$

PROBLEM TWENTY FOUR:

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^4$?

Prove or provide a counter example.
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PROBLEM TWENTY FIVE:
Write the linear system of equations for the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

How many solutions does the system have?

PROBLEM TWENTY SIX:
Consider the linear system

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 0 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \\ 7 \end{bmatrix}.$$

Task: Solve the system by performing the following steps:

1. **Find a particular solution** x_p such that $Ax_p = b$.
2. **Solve the homogeneous system** $Ax = 0$ to find the general solution x_h.
3. **Express the general solution** of the system in parametric vector form.

PROBLEM TWENTY SEVEN:
Problem: For each set of vectors below, determine whether the set is linearly independent or linearly dependent. Provide a justification for your answer.

(a) Set of 4 vectors in $\mathbb{R}^3$:
Let

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(b) Set of 3 vectors in $\mathbb{R}^3$:
Let

$$T = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Instructions: For each set, explain your reasoning and describe how you determined whether the set is linearly independent or dependent. You may use methods such as:

- Forming a matrix with the vectors as columns and using row reduction to check for pivot positions.
- Directly applying the definition of linear independence.

PROBLEM TWENTY EIGHT:

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(1,0) = (0,1), \quad T(0,1) = (2,0), \quad T(1,1) = (1,1).$$

Determine whether T is a linear transformation. If T is linear, find the matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$. If T is not linear, demonstrate that it is not linear.

PROBLEM TWENTY NINE:
Consider the following homogeneous system of linear equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0, \\ 2x_1 + 4x_2 - 2x_3 + 2x_4 &= 0, \\ x_1 + x_2 + x_3 - x_4 &= 0. \end{aligned}$$

(a) Write the corresponding coefficient matrix.

(b) Does this system have any non-trivial solutions? Justify your answer.

(c) Suppose we set $x_3 = -1$. Is the resulting system still consistent? Explain why or why not, and how you know.

PROBLEM THIRTY:

Describe the solution set of the system

$$\begin{array}{rrrl} -x_1 & +3x_2 & -2x_3 & = 0 \\ -4x_1 & -2x_2 & +2x_3 & = 0 \\ x_1 & +4x_2 & -3x_3 & = 0 \end{array}$$

in parametric vector form. Also, give a geometric description PROBLEM THIRTY ONE:

Describe the solution set of the system

$$\begin{array}{rrrl} -x_1 & +3x_2 & -2x_3 & = -1 \\ -4x_1 & -2x_2 & +2x_3 & = 2 \\ x_1 & +4x_2 & -3x_3 & = -2 \end{array}$$

Also, give a geometric description of the solution set and compare it to that in part a

PROBLEM THIRTY TWO:

Are the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ linearly independent? If yes, write the vector $b = \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$ as a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$. PROBLEM THIRTY

THREE:

Is the system

$$\begin{array}{rrrl} x_1 & +2x_2 & -3x_3 & = -1 \\ -2x_1 & -4x_2 & +5x_3 & = -2 \\ -x_1 & -2x_2 & +8x_3 & = -3 \\ -2x_1 & -4x_2 & -6x_3 & = 3 \end{array}$$

consistent. If it is consistent, determine the solution set. If it is not consistent, explain why.

PROBLEM THIRTY FOUR:

Is $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 \\ x_2 \end{bmatrix}$. Is T a linear transformation? If it is, find the standard matrix for T. If it is not a linear transformation, explain why.