Test 4 — Math 203 — November 25th 2024
CLOSED-BOOK — ATTEMPT ALL QUESTIONS
TIME ALLOWED: 1 HOUR

Calculators and notes are not allowed. To obtain full credit you should show all your working clearly and concisely. You may attempt the questions in any order. All questions are worth equal marks.

Question 1

a) Define what is meant by an *eigenvalue* and *eigenvector* of a matrix A.

b) The *Cayley-Hamilton Theorem* states that any matrix satisfies its own characteristic equation (i.e. if the characteristic polynomial is $p(\lambda)$, then $p(A) = 0$). Verify the theorem for the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

a) λ is an e-value of A if $\exists\, \underline{x} \neq \underline{0}$ s.t. $A\underline{x} = \lambda \underline{x}$

$\underline{x}$ is an e-vector associated with λ.

b) $p(\lambda) = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$

$p(A) = A^2 - 5A - 2I$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$5A = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}, \quad 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A^2 - 5A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

Question 2 Find the eigenvalues and eigenspaces of the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

and a diagonalization of A if possible.

Go down 3^{rd} column. $\det(A-\lambda I) = (1-\lambda)(3-\lambda)(1-\lambda)$

So e-values are $\lambda = 1, 1, 3$

$\underline{\lambda = 3}$ $\quad A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 4 & 2 & -2 \end{pmatrix}$. By inspection (or solving $(A-\lambda I)\underline{x} = 0$)

eigenspace $= \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$.

$\underline{\lambda = 1}$ $\quad A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_2, x_3 free $\quad x_1 = -\frac{1}{2}x_2$. So e-space $= \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right\}$ or similar

So $A = PDP^{-1}$ where $P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 3 \end{pmatrix}$

(orders may be different but columns of P must correspond to the correct e-value).

Question 3

Prove each true statement or find a counterexample for each false one.

a) Let A be a 3×3 matrix with eigenvalues 2,3 and 4. Then $\det(A) = 9$.

b) Let A be a 3×3 diagonalizable matrix, such that A^2 has eigenvalues 1,4 and 9. Then A has eigenvalues 1,2 and 3.

c) There are at least two 2×2 matrices, with eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ corresponding to the respective eigenvalues -2 and 0.

d) There is no matrix A that has the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ as eigenvectors corresponding, respectively, to the eigenvalues of 1,2 and 3.

a) False $A = \begin{pmatrix} 2 & & \\ & 3 & 0 \\ 0 & & 4 \end{pmatrix}$ $\det A = 2 \times 3 \times 4 = 12$

b) False $A = \begin{pmatrix} -1 & & \\ & -2 & 0 \\ 0 & & -3 \end{pmatrix}$ for example

c) False Such a matrix equals PDP^{-1}

where either $P = \begin{pmatrix} 1 & -3 \\ 2 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$

or $P = \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$

but these are the same.

d) True. The e-values are distinct but the e-vectors are not lin. ind

($\underline{v}_3 = \underline{v}_1 + \underline{v}_2$).

Question 4 Define $T : \mathbb{P}_3 \to \mathbb{P}_3$ by $T(\mathbf{p}) = \mathbf{p}(0) - \mathbf{p}(1)t - \mathbf{p}(1)t^2 + \mathbf{p}(0)t^3$. Find $T(\mathbf{p})$ when $T(\mathbf{p}) = t + t^2$. Is $\mathbf{p}$ an eigenvector of T? If so what is the eigenvalue?

Using the standard basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$

T is represented by the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$p = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\quad A\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -2 \\ 0 \end{pmatrix}$.

So p is an e-vector of T with e-value -2