Test 3 — Math 203 — November 6th 2024
CLOSED-BOOK — ATTEMPT ALL QUESTIONS
TIME ALLOWED: 1 HOUR

Calculators and notes are not allowed. To obtain full credit you should show all your working clearly and concisely. You may attempt the questions in any order. All questions are worth equal marks.

Question 1 Find bases for the null space, column space and row space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Pivot columns are 1 and 2

Basis for col(A) = columns 1 and 2 of A $= \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$

Basis for row(A) = non-zero rows of B $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$$B \sim \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3 x_4 free $\quad x_2 = x_4 - x_3$
$\quad x_1 = -x_3 - x_4$

So G.S. of $A\underline{x} = \underline{0}$ is $x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

So basis for nul(A) is $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Question 2 Let B be the basis of $\mathbb{R}^3$ consisting of the vectors $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$ where
$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Given $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find $[\mathbf{x}]_B$ by defining and calculating the appropriate change-of-basis matrix.

Finally, check that your answer for $[\mathbf{x}]_B$ is correct by direct calculation.

$$P_B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad [\underline{x}]_B = P_B^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$P_B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{any way they like!}$$

$$P_B^{-1} \underline{x} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

To check $\quad 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \underline{x}$ ✓

Question 3

a) Define the **null space,** nul(A), of an $m \times n$ matrix A. Prove that the null space is a subspace of $\mathbb{R}^n$.

b) Prove that if there exists a basis of a vector space V that has exactly n elements then any set of p vectors with $p > n$ must be linearly dependent.

a) nul (A) = all $\underline{x} \in \mathbb{R}^n$ s.t. $A\underline{x} = \underline{0}$

$A\underline{0} = \underline{0}$ so $\underline{0} \in$ nul (A).

if $\underline{u} \in$ nul A then $A(r\underline{u}) = rA(\underline{u}) = \underline{0}$ so $r\underline{u} \in$ nul (A) $\forall r \in \mathbb{R}$

if $\underline{u}, \underline{v} \in$ nul A then $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$ so $\underline{u} + \underline{v} \in$ nul (A) ✓

b) Theorem 10. Section 4.5 See book.

Question 4 Show that the polynomials $1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3$ form a basis for $\mathbb{P}_3$. Also, find the coordinate vector of the polynomial $p(t) = 14 - 16t + 6t^2$.

Using the standard basis for $\mathbb{P}_3$, the matrix P_B is $\begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Pivot in every row and column. So spans and is lin. ind. $\Rightarrow$ basis.

Backsolving $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 6 & 14 \\ 0 & -1 & -4 & -18 & -16 \\ 0 & 0 & 1 & 9 & 6 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$

gives $x_4 = 0$, $x_3 = 6$, $x_2 = -16 + 24 = 8$

$x_1 = 14 - 12 - 8 = -6$

So $[p_t]_B = \begin{pmatrix} -6 \\ 8 \\ 6 \\ 0 \end{pmatrix}$

Question 5 Solve the linear system

$$\begin{aligned} x+y &= 2 \\ 2x+3y &= 4 \end{aligned}$$

using Cramer's Rule.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\det A = 3 - 2 = 1$$

$$\det A_1(\underset{\sim}{b}) = \det \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = 2$$

$$\det (A_2(b)) = \det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0$$

So $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}}$