§6.1 HW Solutions

2) $\underset{\sim}{w} \cdot \underset{\sim}{w} = 35$, $\underset{\sim}{x} \cdot \underset{\sim}{w} = 5$ $\quad \dfrac{\underset{\sim}{x} \cdot \underset{\sim}{w}}{\underset{\sim}{w} \cdot \underset{\sim}{w}} = \dfrac{5}{35} = \dfrac{1}{7}$

10) $\left\| \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} \right\| = \sqrt{3^2 + 6^2 + (-3)^2} = \sqrt{54}$. So a unit vector is $\dfrac{1}{\sqrt{54}} \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} = \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

14) $\underset{\sim}{u} - \underset{\sim}{z} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$ and $\left\| \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} \right\| = \sqrt{16 + 16 + 4} = 6$

19) – 28) T T T T T T F T F T

32) $\| \underset{\sim}{u} + \underset{\sim}{v} \|^2 = (\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{u} + 2 \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v} = \|\underset{\sim}{u}\|^2 + 2 \underset{\sim}{u} \cdot \underset{\sim}{v} + \|\underset{\sim}{v}\|^2$

$\| \underset{\sim}{u} - \underset{\sim}{v} \|^2 = (\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{u} - 2 \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v} = \|\underset{\sim}{u}\|^2 - 2 \underset{\sim}{u} \cdot \underset{\sim}{v} + \|\underset{\sim}{v}\|^2$

Add the two equations.

34) W is the nullspace of the matrix $(5 \; -6 \; 7)$ so by Theorem 2 of Chapter 4 it is a subspace. It is a plane through 0.

37) Let $\underset{\sim}{w} = c_1 \underset{\sim}{v}_1 + \ldots + c_p \underset{\sim}{v}_p$ be any vector in W

Then $\underset{\sim}{w} \cdot \underset{\sim}{x} = (c_1 \underset{\sim}{v}_1 + \ldots + c_p \underset{\sim}{v}_p) \cdot \underset{\sim}{x} = c_1 (\underset{\sim}{v}_1 \cdot \underset{\sim}{x}) + \ldots + c_p (\underset{\sim}{v}_p \cdot \underset{\sim}{x}) = 0$

Since $\underset{\sim}{x}$ is orthogonal to every element in the basis.

39) $\underset{\sim}{x} \in W^{\perp} \Rightarrow \underset{\sim}{x}$ is orthogonal to every element in W.

But $\underset{\sim}{x} \in W$ so $\underset{\sim}{x} \cdot \underset{\sim}{x} = 0 \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}$

§6.2 Homework Solutions

1) No (1st and 3rd not orthogonal) 2) Yes

7) $\underline{u}_1 \cdot \underline{u}_2 = 0$ so $\underline{x} = \frac{\underline{x}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\underline{u}_1 + \frac{\underline{x}\cdot\underline{u}_2}{\underline{u}_2\cdot\underline{u}_2} = \frac{39}{13}\underline{u}_1 + \frac{26}{52}\underline{u}_2$

$= 3\underline{u}_1 + \frac{1}{2}\underline{u}_2$

8) Same as 7). $\underline{x} = -\frac{9}{10}\underline{u}_1 + \frac{13}{20}\underline{u}_2$

12) Let $\underline{y} = \begin{pmatrix}1\\-1\end{pmatrix}$, $\underline{u} = \begin{pmatrix}-1\\2\end{pmatrix}$ $\quad \text{proj}_{\underline{u}}\,\underline{y} = \frac{\underline{y}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\underline{u} = \begin{pmatrix}0.6\\-1.2\end{pmatrix}$

15) $\|\underline{y} - \hat{\underline{y}}\| = \left\| \begin{pmatrix}3\\1\end{pmatrix} - \frac{\underline{y}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\underline{u} \right\| = 1$

20) $\begin{pmatrix}-2/3\\1/3\\2/3\end{pmatrix} \cdot \begin{pmatrix}1/3\\2/3\\0\end{pmatrix} = 0$ so orthogonal. Normalizing $\Rightarrow \begin{pmatrix}-2/3\\1/3\\2/3\end{pmatrix}$ and $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix}$

23) T 24) T (one of the vectors might be $\underline{0}$) 25) T

26) F (also need $\underline{u}_i \cdot \underline{u}_i = 1$) 27) F 28) T

29) F (must be square) 30) T 31) F ($\underline{y} - \hat{\underline{y}}$ does) 32) T

§6.3 Homework Solutions

1) See 2) and answer in back of book.

2) $\underline{v} \cdot \underline{u}_1 = 14$ so $\underline{v} = 14\underline{u}_1 + \underbrace{(\underline{v} - 14\underline{u}_1)}_{\text{in span } \{\underline{u}_2, \underline{u}_3, \underline{u}_4\}}$.

4) $\underline{u}_1 \cdot \underline{u}_2 = 0$ ✓ $\quad \text{proj}_W \underline{y} = \dfrac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \dfrac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2$

$= \dfrac{24}{25}\underline{u}_1 + \dfrac{7}{25}\underline{u}_2 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ [or spot that span $\{\underline{u}_1, \underline{u}_2\}$ is the x-y plane!]

8) Since $\underline{u}_1 \cdot \underline{u}_2 = 0$ $\quad \hat{\underline{y}} = \text{proj}_W \underline{y} = \dfrac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} + \dfrac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} = \begin{pmatrix} 3/2 \\ 7/2 \\ 1 \end{pmatrix}$

$\underline{z} \in W^\perp$ is given by $\underline{y} - \hat{\underline{y}} = \begin{pmatrix} -5/2 \\ 1/2 \\ 2 \end{pmatrix}$

14) $\hat{\underline{z}} = \frac{7}{14}\underline{u}_1 + 0\,\underline{u}_2 = \frac{1}{2}\underline{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{pmatrix}$

19) $\underline{u}_3 - \text{proj}_W \underline{u}_3 = \begin{pmatrix} 0 \\ 2/5 \\ 1/5 \end{pmatrix}$ is such a vector.

21) T 22) T 23) F 24) T 25) F (it is $\text{proj}_W \underline{y}$)

26) T 27) T 28) T 29) T 30) F (unless $n=p$)

§6.4 Homework Solutions

3) $\underline{v}_1 = \underline{x}_1, \quad \underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix} \left(\text{or } \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right)$

7) $\frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

10) $\underline{v}_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 6 \\ -8 \\ -2 \\ 4 \end{pmatrix} - \frac{-36}{12}\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

$\underline{v}_3 = \begin{pmatrix} 6 \\ 3 \\ 6 \\ -3 \end{pmatrix} - \frac{6}{12}\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} - \frac{30}{12}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}.$

17) F (c might be 0) 18) F (they must not be 0)

19) T 20) T 21) T (see Example 4) 22) T (Theorem 12)

§6.5 Solutions

1)

Answers for Homework questions from Section 6.5

Question 1 The normal equations are $(A^TA)\mathbf{x} = A^T\mathbf{b}$ and are given by

$$\begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 11 \end{pmatrix}.$$

The solution is $\hat{\mathbf{x}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Question 3 The normal equations are

$$\begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

and their solution is $\hat{\mathbf{x}} = \begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix}$.

True or False
17) True, by the definition of least squares solution.
18) True.
19) True.
20) False. The inequality is reversed.
21) True. See Theorem 13.
22) True. See Theorem 14.
23) False. $A\hat{\mathbf{x}}$, not $\hat{\mathbf{x}}$, is the point in col(A) closest to $\mathbf{b}$.
24) True. This is just what it means to solve the linear system.
25) False. Not true if the normal equations are 'ill-conditoned'.
26) False. Calculating R^{-1} is in general very expensive. It's nearly always cheaper to solve the linear system given in equation 7 by using row operations.