Math 203 EXAM 1 Review

Sections covered: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.9

LEARNING OUTCOMES:

- 1- Recognize a system of linear equations
- 2- Reduce a system into echelon form using row operations
- 3- Interpret the solution(s) of a system of linear equations
- 4- Solve a system using backward substitution
- 5- Visualize a matrix-vector product $A\vec{x} = \vec{b}$ as a linear combination of the columns of A, with weights being the entries in $\vec{x}$.
- 6- Understand the relation between the consistency of a system and linear combinations of the columns.
- 7- Compute a matrix-vector product.
- 8- Determine if a homegenous system has a nontrivial solution
- 9- Determine if a set is linearly independent using the solution(s) of a homogeneous system
- 10- Determine if some sets are linearly independent using shortcut rules
- 11- Determine if a homegenous system has a nontrivial solution
- 12- Determine if a set is linearly independent using the solution(s) of a homogeneous system
- 13- Determine if some sets are linearly independent using shortcut rules

PROBLEM ONE: True or False
Mark each statement True or False. Justify each answer or give a counter example.

- a) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent
- b) The columns of a 4 x 5 matrix are linearly dependent.
- c) $(AB)^T = A^T B^T$.
- d) If A is a 2 x 6 matrix and T is a linear transformation defined by $T(\vec{x}) = A\vec{x}$, then the domain of T is $\mathbb{R}^2$.
- e) If A and B are square matrices of the same size then $AB = BA$.

PROBELM TWO: Linear independence

- a)Use a homogeneous system to determine if the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly independent

$$\vec{a_1} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

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- b) Explain how the solution(s) of your homogeneous system determines whether the set above is linearly independent or linearly dependent.

PROBLEM THREE: Matrix of a linear transformation

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - x_2 \\ 2x_2 \end{bmatrix}$

- a) Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$ and let c be any scalar. Prove that T is a linear transformation.
- b) Find the standard matrix A of T.
- c) Is T one-to-one? Prove your answer using the matrix A.
- d) Is T onto? Prove your answer using the matrix A.

PROBLEM FOUR: Matrix Operations
Find $(AB)^T$ given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

PROBLEM FIVE: Solution of a system of linear equations
Find all solutions, if any exist, of the following system. If the system has multiple solutions, write the solution set as the Span of some vectors.

$$x_1 \quad +3x_2 \quad -5x_3 \quad = 0$$

PROBLEM 6

a) Determine whether the following linear system is consistent or not but do not solve it

$$\begin{aligned} 2x_2 + 2x_3 &= 0 \\ x_1 - 2x_4 &= -3 \\ x_3 + 3x_4 &= -4 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5. \end{aligned}$$

b) Row reduce the following matrix to reduced echelon form and list the pivot columns

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

PROBLEM 7 Determine whether or not $A\mathbf{x} = \mathbf{0}$ has any non-trivial solutions where $A = \begin{pmatrix} 1 & 2 & 5 \\ -2 & -3 & -8 \\ -1 & 1 & 1 \end{pmatrix}$.

If it does then write down the solution set.

PROBLEM 8

a) Define precisely when a set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, \ldots \mathbf{a_p}\}$ is *linearly independent.*

b) Prove that if a set of non-zero vectors is linearly dependent then one of the vectors can be written as a linear combination of the previous ones.

PROBLEM 9

a) Consider the linear transformation T that corresponds to the matrix $A = \begin{pmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{pmatrix}$. Find all the vectors that are mapped into the zero vector by T. Does the range of T equal the codomain of T?

b) If instead, A represented the augmented matrix of a linear system of equations, would it be consistent or inconsistent?

PROBLEM 10

For each of the following sets of vectors, determine which values of h make the sets linearly dependent.

a) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 7 \\ -3 \end{pmatrix}, \begin{pmatrix} 8 \\ h \\ 4 \end{pmatrix} \right\}$

Problem 11:
Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 - x_2 + 3x_3 &= -5 \\ -x_2 + x_3 &= 0 \end{aligned}$$

1. Write the augmented matrix.
2. Write the system in vector form.
3. Write the system in matrix form $A\vec{x} = \vec{b}$ (tell us what $A, \vec{x}$, and $\vec{b}$ are for the system above).
4. Is the system consistent? If yes, is the solution unique?

Problem 12:
Let a, b, c, d be real numbers. Assume $a \neq 0$ and $ad - bc \neq 0$. Find the reduced row echelon form of A where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Record the row operations you used and point out where you need to use the assumptions $a \neq 0$ and $ad - bc \neq 0$.

Problem 13:
Mark each statement as True or False.

1. Consider a system of equations $A\vec{x} = \vec{b}$. Let B be the reduced row echelon form (RREF) of A. If every row of B contains a pivot (or leading 1), then the system is guaranteed to have a unique solution.
2. Let A be a matrix and $\vec{v}$ a vector. The product $A\vec{v}$ is well-defined if and only if the number of rows of A equals the number of entries of $\vec{v}$.
3. Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ be a matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Suppose $\vec{a}_3 = -2\vec{a}_1 + 3\vec{a}_2$. Then the equation $A\vec{x} = \vec{0}$ has infinitely many solutions.
4. Let $\vec{a}_1, \vec{a}_2$ be non-zero vectors in $\mathbb{R}^2$. Then $\{\vec{a}_1, \vec{a}_2\}$ is guaranteed to be linearly independent.
5. If $\vec{v}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$, then
$$Span\{\vec{u}_1, \vec{u}_2, \vec{v}\} = Span\{\vec{u}_1, \vec{u}_2\}$$
6. The function
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y - 1 \\ x \end{bmatrix}$$
is a linear transformation.
7. Let A be the matrix of a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$. Suppose that the columns of A span $\mathbb{R}^3$. Then T is onto.

Problem 14:
Let $\vec{b}$ be a vector in $\mathbb{R}^3$ with components b_1, b_2, b_3. Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= b_1 \\ 2x_1 - x_2 + 3x_3 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

2. Describe the set S of all $\vec{b}$ such that the system is consistent by expressing S as the span of a collection of vectors in $\mathbb{R}^3$.
3. Assuming $\vec{b}$ belongs to S, how many free variables appear in the general solution of the original system of equations?

Problem 15:
Let A be a matrix with columns $\vec{a}_1, ..., \vec{a}_k \in \mathbb{R}^n$. Let $\vec{v}$ be a vector with entries $v_1, ..., v_\ell$.

1. What condition must k, ℓ satisfy in order for the product $A\vec{v}$ to be defined?
2. Give the formula for the product $A\vec{v}$ (assuming the condition in (a) holds).
3. Calculate the product $A\vec{v}$ where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

PROBLEM 16:

For all parts of this problem, consider the following system of equations:

$$x_1 + x_3 + 2x_4 = 6$$
$$x_2 - 2x_3 + 5x_4 = 11$$
$$x_1 + 2x_2 - x_3 - 3x_4 = -2$$

1. Rewrite the above system as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$, labeling A, $\mathbf{x}$ and $\mathbf{b}$ clearly.
2. Rewrite the system as a vector equation.
3. Rewrite the system as an augmented matrix.
4. Perform *only* the forward phase of the row reduction algorithm, identifying any pivot positions.
5. How many solutions does the original system have? Are there any solutions at all?
6. Perform the backward phase of the row reduction algorithm on the echelon matrix obtained in part 4, obtaining in the end reduced row echelon form.
7. Identify any free variables and write the other variables in terms of them.
8. Write the solution set in parametric vector form.

PROBLEM 17:

- Determine r and s such that the solution set of the system

$$-4x + ry = 2$$

$$12x + sy = -4$$

(i) is empty (no solution)
(ii) contains unique solution
(iii) contains many solutions.

- Find the value(s) of r for which the augmented matrix represents a consistent linear system

$$\begin{bmatrix} 1 & r & 4 \\ 3 & 6 & 8 \end{bmatrix}.$$

PROBLEM 18:

- Find the general solution of the system whose augmented matrix is

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}.$$

- Write a system of equations equivalent to the vector equations:

  (i)

$$x \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + y \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}.$$

  (ii)

$$x \begin{bmatrix} -2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 8 \\ 5 \end{bmatrix} + z \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

PROBLEM 19:

- Compute the products using row-vector rule for computing Ax. If the product is undefined, explain why.

  (i)

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

  (ii)

$$\begin{bmatrix} 8 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Does the system

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

  has a nontrivial solution?

PROBLEM 20

- State with reason whether the vectors

  (i)
$$\begin{bmatrix}5\\1\end{bmatrix}, \begin{bmatrix}2\\8\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-1\\7\end{bmatrix}$$

  (ii)
$$\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}-2\\5\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$$

  are linearly dependent.

- Let $A = \begin{bmatrix}1 & 0 & -2\\-2 & 1 & 6\\3 & -2 & -5\end{bmatrix}$,, $u = \begin{bmatrix}1\\0\\-4\end{bmatrix}$, and $v = \begin{bmatrix}a\\b\\c\end{bmatrix}$.

  Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = Ax$. Find $T(u)$ and $T(v)$.

PROBLEM 21:

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x_1, x_2) = (x_1 - 2x_2,\ -x_1 + 3x_2,\ 3x_1 - 2x_2).$$

Find x such that $T(x) = (-1, 4, 9)$.

PROBLEM 22:

Let $\mathbf{u} = \begin{bmatrix}-1\\3\\-2\end{bmatrix}$], $\mathbf{v} = \begin{bmatrix}4\\0\\-1\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}-3\\-1\\2\end{bmatrix}$. Compute the indicated vector.

1. $\mathbf{u} + 2\mathbf{v} - 4\mathbf{w}$
2. $4\mathbf{u} - 2\mathbf{v} + 4\mathbf{w}$

PROBLEM 23:

Find all scalars c, if any exist, such that the given statement is true.

1. The vector $\begin{bmatrix}c\\-c\\4\end{bmatrix}$ is a scalar multiple of the vector $\begin{bmatrix}-2\\2\\20\end{bmatrix}$.
2. The vector $\begin{bmatrix}3\\-2\\c\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\3\end{bmatrix}$.

PROBLEM 24:
Describe all solutions of a linear system whose corresponding augmented matrix can be row reduced to the given matrix. Write the solution, if it exists, in vector parametric form.

1.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

2.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & 0 & -4 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

PROBLEM 25:
Find all solutions of the given linear system. If the solution exists, write it in vector parametric form.

1.

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & 2x_3 & = & 4 \\ 2x_1 & + & 7x_2 & - & x_3 & = & -2 \\ 2x_1 & + & 9x_2 & - & 7x_3 & = & 1 \end{array}$$

2.

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 2x_3 & + & x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & - & 3x_4 & = & 0 \\ 9x_1 & - & 3x_2 & - & x_3 & - & 7x_4 & = & 4 \end{array}$$

PROBLEM 26:
Determine whether the columns of the given matrix span $\mathbb{R}^4$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & -1 \end{bmatrix}$$

PROBLEM 27:

Determine whether the given set of vectors is dependent or independent.

1. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

2. $\left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 13 \\ 4 \\ 7 \end{bmatrix} \right\}$ in $\mathbb{R}^4$.

PROBLEM 28:

Find all scalars s, if any exist, such that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ s \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -s \\ 0 \end{bmatrix}$ are linearly independent.

PROBLEM 29:
Show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

1. $T : \mathbb{R}^2 \to \mathbb{R}^3, T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2, x_1 + 3x_2)$
2. $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x_1, x_2, x_3) = (x_1 - x_2 + 3x_3, x_1 + x_2 + x_3, x_1)$

PROBLEM 30:

1. If $T(1, 0) = (3, -1)$ and $T(0, 1) = (-2, 5)$, Find $T(4, -6)$.
2. If $T(-1, 2) = (1, 0, 0)$ and $T(2, 1) = (0, 1, 2)$, Find $T(0, 10)$.

PROBLEM 31:
Give the matrix representation of the rotation of the plane counterclockwise about the origin through an angle of

1. $30°$
2. $90°$
3. $135°$

PROBLEM 32:
Show that the linear transformation

$$T(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1. A vertical expansion, if $r > 1$
2. A vertical expansion followed by a reflection in the x–axis if $r < -1$.