Math 203 Fall 2024 Test 2 Solutions

1) $\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 7 & 0 & 1 & 0 \\ -3 & 6 & -2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -13 & 1 & -3 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -28 & 2 & -7 \\ 0 & 1 & 0 & -13 & 1 & -3 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right)$. So $A^{-1} = \underline{\begin{pmatrix} -28 & 2 & -7 \\ -13 & 1 & -3 \\ 3 & 0 & 1 \end{pmatrix}}$

Find $\det(A^{-1})$ first since A^{-1} has a zero.

$$\det A^{-1} = \begin{vmatrix} -28 & 2 & -7 \\ -13 & 1 & -3 \\ 3 & 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & -7 \\ 1 & -3 \end{vmatrix} + 1 \begin{vmatrix} -28 & 2 \\ -13 & 1 \end{vmatrix} = 3 + (-2) = \underline{\underline{1}}$$

$$\det A = \frac{1}{\det A^{-1}} = \frac{1}{1} = \underline{\underline{1}}$$

2) Eg go down second column.

$$\text{determinant} = (-3) \begin{vmatrix} 1 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & -1 & 2 \end{vmatrix} = (-3)(2) \begin{vmatrix} 1 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$$

$= (-3)(2) \begin{vmatrix} 1 & 3 & -5 \\ 0 & -13 & 22 \\ 0 & -1 & 2 \end{vmatrix}$ by subtracting $5 \times$ row 1 from row 2

$$= -6 \begin{vmatrix} -13 & 22 \\ -1 & 2 \end{vmatrix} = (-6)(-26+22) = \underline{\underline{24}}$$

3) a) $A^2 = A \Rightarrow \det A^2 = \det A \Rightarrow (\det A)(\det A) = \det A \Rightarrow \underline{\det A = 0 \text{ or } 1}$

b) $A^n = A \ \forall n > 2$. Either Induction if you know it or

$A^3 = A^2 A = AA = A^2 = A$

$A^4 = A^3 A = AA = A^2 = A$

etc.

c) For example $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Identity}}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Projections}}$

4) $\det B = -(-1)\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} + 1 \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} = 6 + 6 = 12 \neq 0$ so B^{-1} exists

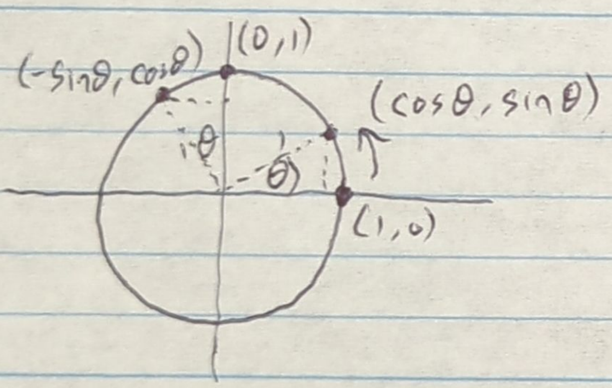
$$(AB^{-1})^{-1} = BA^{-1} = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -3 & 5 \\ 10 & 0 & 10 \\ -3 & 1 & 1 \end{pmatrix}$$

AB^{-1} ~~This~~ has more fractions than I intended!

The answer is $\frac{1}{30}\begin{pmatrix} -5 & 4 & -15 \\ 20 & 14 & -30 \\ 5 & -1 & 15 \end{pmatrix}$

5) a) T is linear if a) $T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v}) \quad \forall\, \underline{u}, \underline{v} \in \mathbb{R}^n$

b) $T(c\underline{u}) = cT(\underline{u}) \quad \forall\, c \in \mathbb{R},\ \underline{u} \in \mathbb{R}^n$.

b) 

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

So matrix is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

c) $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cancel{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$