A SHORT MATLAB IMPLEMENTATION OF FRACTIONAL POISSON EQUATION WITH NONZERO BOUNDARY CONDITIONS *

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Abstract. The purpose of this note is to provide a standalone Matlab code to solve fractional Poisson equation with nonzero boundary conditions based on Antil, Pfefferer, Rogovs [1] [1]. We will use the approach of Bonito and Pasciak [5] to solve the fractional Poisson equation with zero boundary conditions. The discretization is carried out using piecewise linear finite element method. Remarkably enough it is sufficient to have access to the mass and the stiffness matrix for standard Laplacian to solve such fractional Poisson problems.

All the mathematical results discussed here are already available in the literature.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be an open, bounded domain with boundary $\partial\Omega$. For a given f and g we seek u solving the following system

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $(-\Delta)^s$ denotes the fractional powers of Laplacian with nonzero Dirichlet boundary condition. This operator was first defined in [1]

$$(-\Delta)^s u := \sum_{k=1}^{\infty} \left(\lambda_k^s u_{\Omega,k} + \lambda_k^{s-1} u_{\partial\Omega,k} \right) \varphi_k, \tag{1.2}$$

with $u_{\Omega,k} = \int_\Omega u\varphi_k$ and $u_{\partial\Omega,k} = \int_{\partial\Omega} u \partial_\nu \varphi_k$. If $u|_{\partial\Omega} = 0$ we let $(-\Delta_0)^s := (-\Delta)^s$, i.e., fractional Laplacian with zero Dirichlet boundary conditions. Here λ_k, φ_k are the eigenvalues and eigenvectors for the standard Laplacian with zero boundary conditions, i.e.,

$$-\Delta\varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega, \quad \varphi_k = 0 \quad \text{on } \partial\Omega.$$

Under appropriate regularity assumptions on f, g, and Ω existence and uniqueness of weak solution to (1.1) is given in [1, Theorem 4.5]. In fact the paper [1] discusses other boundary conditions such as Neumann and the approach presented there directly extends to more general boundary conditions such as Robin or mixed boundary conditions.

To proceed further, we employ the lifting argument as in [1]. We begin by writing u solving (1.1) as

$$u = w + v_g, \tag{1.3}$$

where $w|_{\partial\Omega} = 0$ and v_g is the lifting of the boundary datum g to the domain Ω that fulfills $v_g|_{\partial\Omega} = g$. A particular choice of such a v_g is the s-harmonic lifting, i.e.,

$$(-\Delta)^s v_g = 0 \quad \text{in } \Omega, \quad v_g = g \quad \text{on } \partial\Omega. \tag{1.4}$$

Thus w fulfills

$$(-\Delta_0)^s w = f \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

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[1]The code is available at https://harbirantil@bitbucket.org/harbirantil/frac_poisson_nhbc.git

Whence to solve (1.1) we need to solve (1.4) and (1.5). Indeed (1.5) can be solved using the existing techniques such as the Dunford-Taylor approach [7, 5, 6]. However, (1.4) is challenging. In [1] we proved that (1.4) is equivalent to v_g being harmonic in the very-weak sense

$$-\Delta v_g = 0 \quad \text{in } \Omega, \quad v_g = g \quad \text{on } \partial\Omega. \tag{1.6}$$

Here by very weak sense, we mean

$$\int_\Omega v_g(-\Delta)\varphi \, \mathrm{d}x = -\int_{\partial\Omega} g \partial_\nu \varphi \, \mathrm{d}s \quad \forall \varphi \in \mathrm{dom}(-\Delta) \cap H_0^1(\Omega). \tag{1.7}$$

Notice that in case $g \in H^{\frac{1}{2}}(\partial\Omega)$ a very weak solution is just the weak solution, i.e., $v_g \in \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\}$ such that

$$\int_\Omega \nabla v_g \cdot \nabla \varphi \, \mathrm{d}x = 0 \quad \forall \varphi \in H_0^1(\Omega). \tag{1.8}$$

Thus to obtain u we are reduced to solving (1.5) and (1.7) (and in case g is smooth enough (1.8)).

Remark 1.1 (s-harmonic extension vs general case). Notice that the approach discussed in [1] is not limited to the splitting as in (1.3) with v_g being s-harmonic but it can directly applied to (1.1) using the integration by parts formula [1, Theorem 3.3]. We will describe this in Section 3.

Remark 1.2 (General second order elliptic operators). The approach discussed above directly applies to the case when one is interested in powers of general second order elliptic operators for instance L^s with

$$Lw = -\mathrm{div}(A\nabla w) + cw,$$

where $A = (a_{ij})_{i,j=1}^N$ with a_{ij} measurable and belonging to $L^\infty(\Omega)$, is symmetric and satisfy the ellipticity condition. Moreover, $0 \leq c \in L^\infty(\Omega)$.

2. Finite element discretization. We next present a discretization for (1.7) and (1.5). We will assume that Ω is polygonal or polyhedral.

2.1. Discretization of very weak formulation. Let $\mathcal{T}_\Omega$ be a conforming, quasi-uniform triangulation of Ω

$$V_h := \{v_h \in C^0(\bar{\Omega}) : v_h|_T \in \mathcal{P}_1 \, \forall T \in \mathcal{T}_\Omega\}, \; V_{0h} := V_h \cap H_0^1(\Omega), \; V_h^\partial := V_h|_{\partial\Omega}$$

and

$$V_{*h} := \{v_h \in V_h : v_h|_{\partial\Omega} = \Pi_h g\},$$

where $\Pi_h g \in V_h^\partial$ denotes the L^2-projection of g. Notice that this discretization is valid for g which are less regular than $H^{\frac{1}{2}}(\partial\Omega)$.

The discrete problem is given by: Find $v_h \in V_{*h}$ such that

$$\int_\Omega \nabla v_h \cdot \nabla \varphi_h \, \mathrm{d}x = 0 \quad \forall \varphi_h \in V_{0h}, \tag{2.1}$$

see [4, 2, 3] for further details. Notice that with the above discretization we are solving the classical Poisson equation with nonzero boundary conditions. There are various ways of imposing nonzero Dirichlet boundary conditions the above approach is just one of them that allows functions g which are even less regular than $H^{\frac{1}{2}}(\Omega)$ with the help of Π_h.

2.2. Discretization of fractional Poisson problem with zero boundary conditions.

Continuous problem. In order to realize (1.5) the authors in [5] used the *Balakrishnan formula* [7]

$$(-\Delta)^{-s}\zeta = \frac{\sin(s\pi)}{\pi}\int_0^\infty \mu^{-s}(\mu-\Delta)^{-1}\zeta d\mu.$$

Going back to the equation (1.5) and using the above formula we obtain that

$$\begin{aligned} w(x) = (-\Delta)^{-s}f(x) &= \frac{\sin(s\pi)}{\pi}\int_0^\infty \mu^{-s}(\mu-\Delta)^{-1}f(x)d\mu \\ &= \frac{\sin(s\pi)}{\pi}\int_{-\infty}^\infty e^{(1-s)y}(e^y-\Delta)^{-1}f(x)dy \end{aligned}$$

where in the last step we have used the change of variables $\mu = e^y$. After setting

$$(-\Delta + e^y)z = f \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

we arrive at

$$w(x) = \frac{\sin(s\pi)}{\pi}\int_{-\infty}^\infty e^{(1-s)y}z(y)dy. \tag{2.3}$$

Thus to find w for each y we need to solve (2.2) and then calculate w by computing the integral in (2.3).

Discrete problem. We follow [5]: We write a discretization for (2.2) in space and we use sinc quadrature to approximate the integral, i.e., let $z_h(y_\ell) \in V_{0h}$ solve

$$\int_\Omega (\nabla z_h \cdot \nabla v_h + e^{y_\ell} z_h v_h)\,\mathrm{d}x = \int_\Omega f v_h\,\mathrm{d}x \quad \text{in } \Omega, \quad \text{for all } v_h \in V_{0h} \tag{2.4}$$

then

$$w_h = k\frac{\sin(s\pi)}{\pi}\sum_{\ell=-N_-}^{N_+} e^{(1-s)y_\ell} z_h(y_\ell), \tag{2.5}$$

where $N_+ = \lceil \frac{\pi^2}{4sk^2} \rceil$, $N_- = \lceil \frac{\pi^2}{4(1-s)k^2} \rceil$, $y_\ell = k\ell$, $k = 1/\log(1/h)$.

Thus to approximate w in (2.3) we are reduced to solving (2.4) for each quadrature point y_ℓ. These solves are completely independent of each other.

Finally we obtain approximation of u as

$$u_h = v_h + w_h$$

where v_h and w_h are given by (2.1) and (2.5), respectively.

3. Integration by parts formula. Applying a test function ζ to (1.1) and using the integration by parts formula from [1, Theorem 3.3] we arrive the following form of (1.1)

$$\int_\Omega u(-\Delta_0)^s\zeta\,\mathrm{d}x = \int_\Omega f\zeta\,\mathrm{d}x - \int_{\partial\Omega} g\partial_\nu z_\zeta\,\mathrm{d}s \quad \forall \zeta \in \mathrm{dom}((-\Delta_0)^s) \tag{3.1}$$

where z_ζ solves $(-\Delta_0)^{1-s}z_\zeta = \zeta$ in Ω and $z_\zeta = 0$ on $\partial\Omega$.

Another way to rewrite (3.1) is by first letting

$$u = w + v_g$$

where $w|_{\partial\Omega} = 0$ and $v_g|_{\partial\Omega} = g$. Here v_g is the lifting of the boundary datum g to the domain Ω. Substituting the expression of u in (3.1) we obtain that

$$\int_\Omega w(-\Delta_0)^s \zeta \, \mathrm{d}x = \int_\Omega f\zeta \, \mathrm{d}x - \left(\int_\Omega v_g (-\Delta_0)^s \zeta \, \mathrm{d}x + \int_{\partial\Omega} g \partial_\nu z_\zeta \, \mathrm{d}s \right) \ \forall \zeta \in \mathrm{dom}((-\Delta_0)^s). \tag{3.2}$$

Notice by the integration by parts formula of [1, Theorem 3.3] the last term in (3.2) is simply

$$\left(\int_\Omega v_g (-\Delta_0)^s \zeta \, \mathrm{d}x + \int_{\partial\Omega} g \partial_\nu z_\zeta \, \mathrm{d}s \right) = \int_\Omega (-\Delta)^s v_g \zeta \, \mathrm{d}x.$$

One way to evaluate $\int_\Omega (-\Delta)^s v_g \zeta \, \mathrm{d}x$ in practice is by using (1.4) so that v_g is s-Harmonic. Recall that solving (1.4) is equivalent to (1.7).

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