CONTENTS

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WHAT IS A DIGITAL TWIN ?

AIAA Definition:

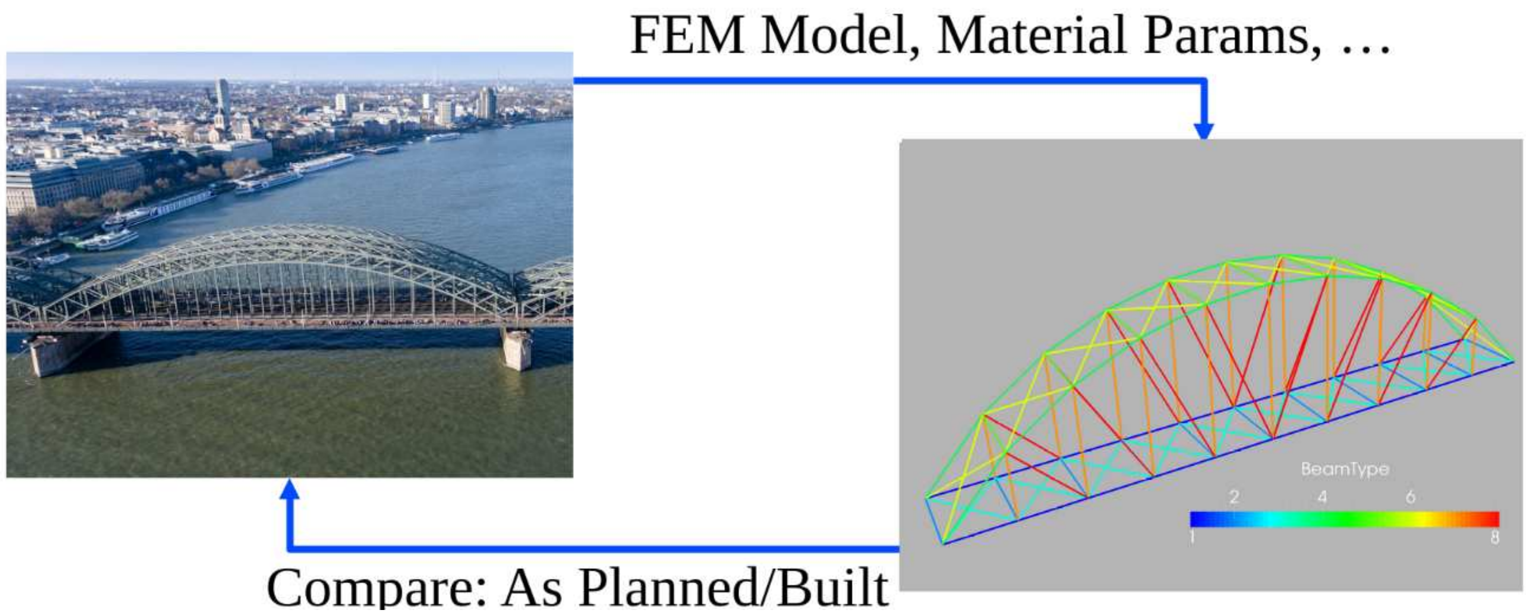
A Digital Twin is a set of virtual information constructs that mimics the structure, context, and behavior of an individual/unique physical asset, is dynamically updated with data from its physical twin throughout its lifecycle, and informs decisions that realize value.

American Institute of Aeronautics and Astronautics (AIAA), Digital Engineering Integration Committee. Digital Twin: Definition and Value. AIAA and AIA Position Paper, 2020. https://www.aiaa.org/docs/default-source/uploadedfiles/issues-and-advocacy/ policy-papers/ digital-twin-institute-position-paper-(december-2020).pdf.

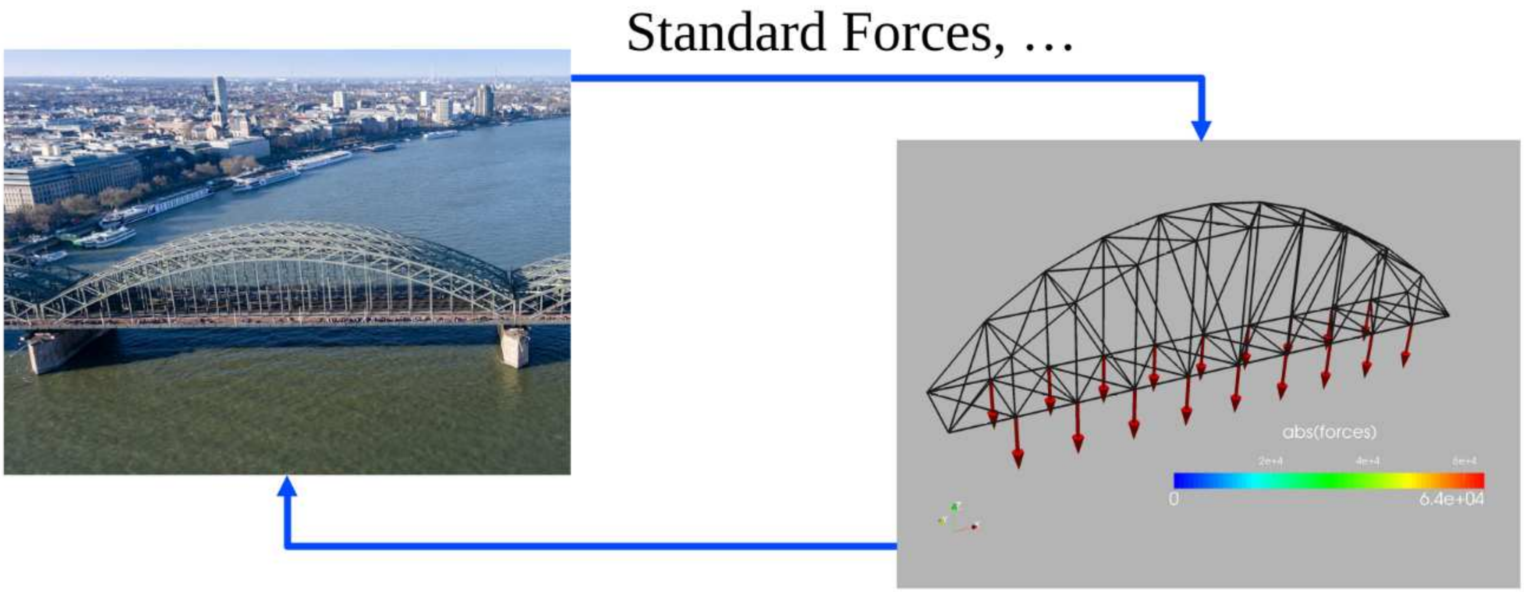
WHAT IS A DIGITAL TWIN ?

- Real ‘Thing’: Object, Process, Patient, ...
- Digital Copy/Mirror
- Sensors
- Infer State of ‘Thing’ from Digital Copy + Sensors
- Update Digital Copy Throughout Lifecycle of ‘Thing’

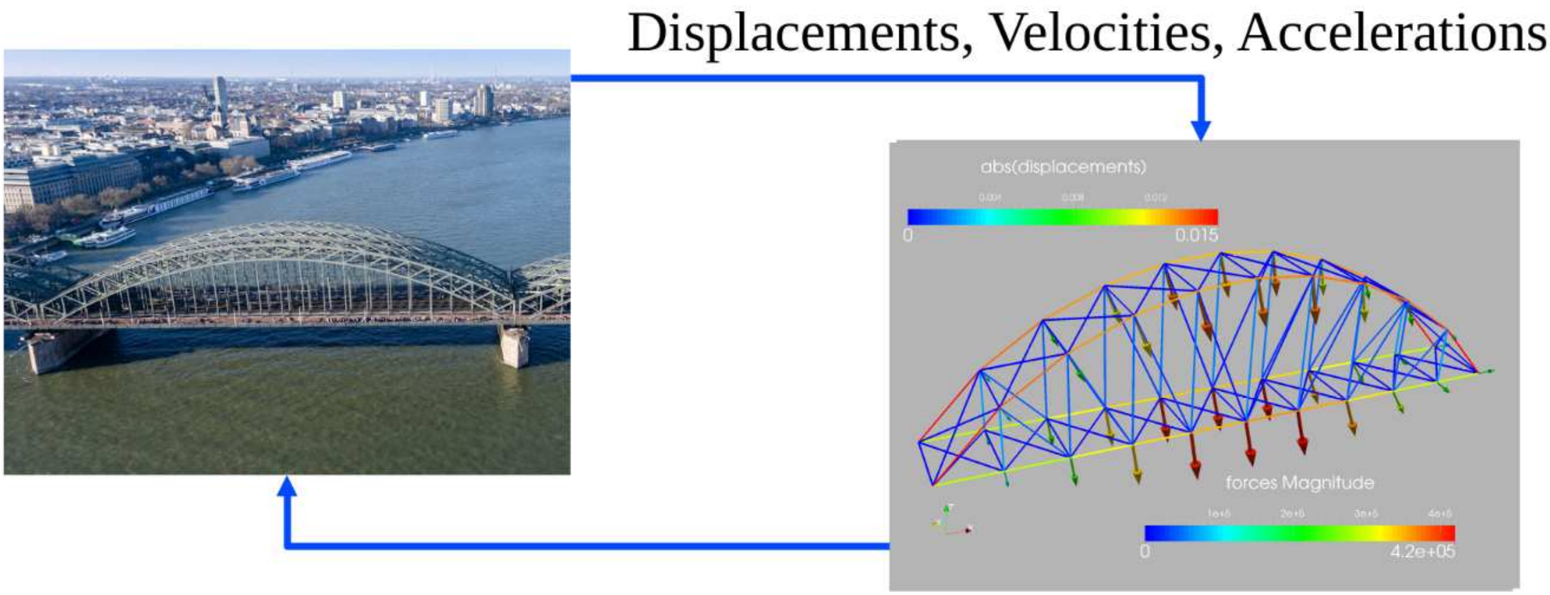
EXAMPLE: BRIDGE (1)



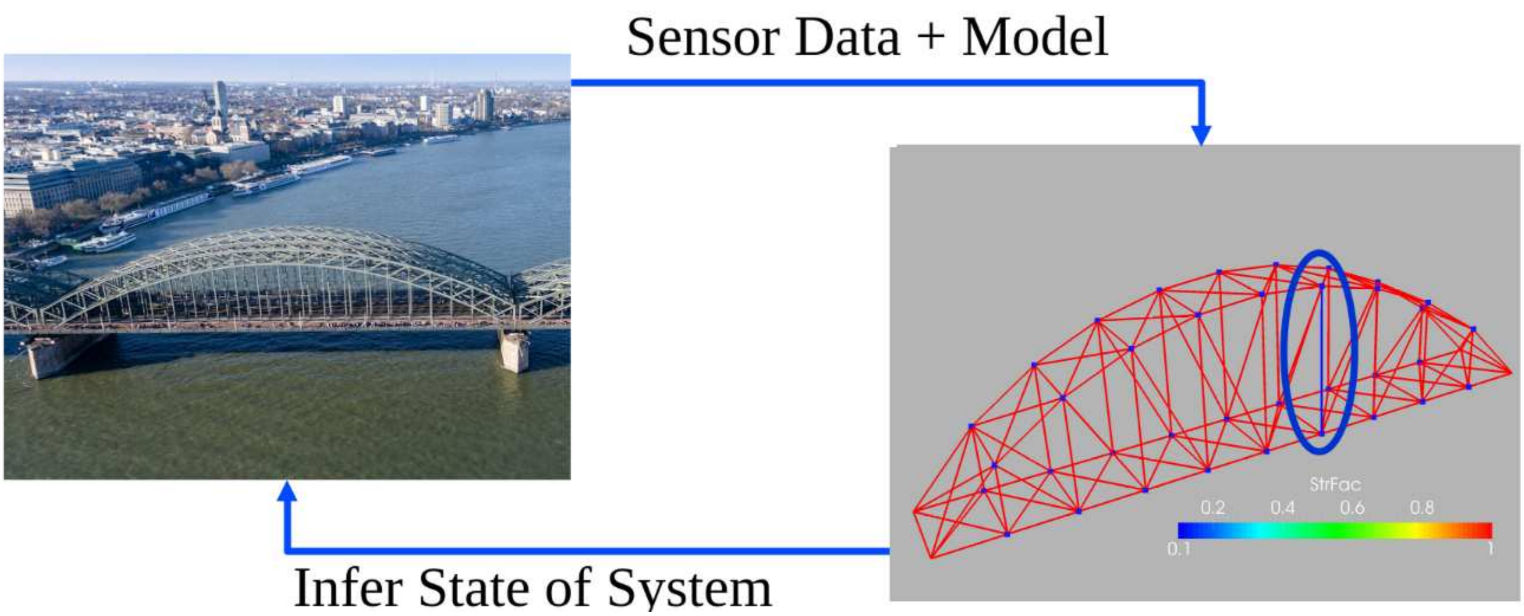
EXAMPLE: BRIDGE (2)



EXAMPLE: BRIDGE (3)

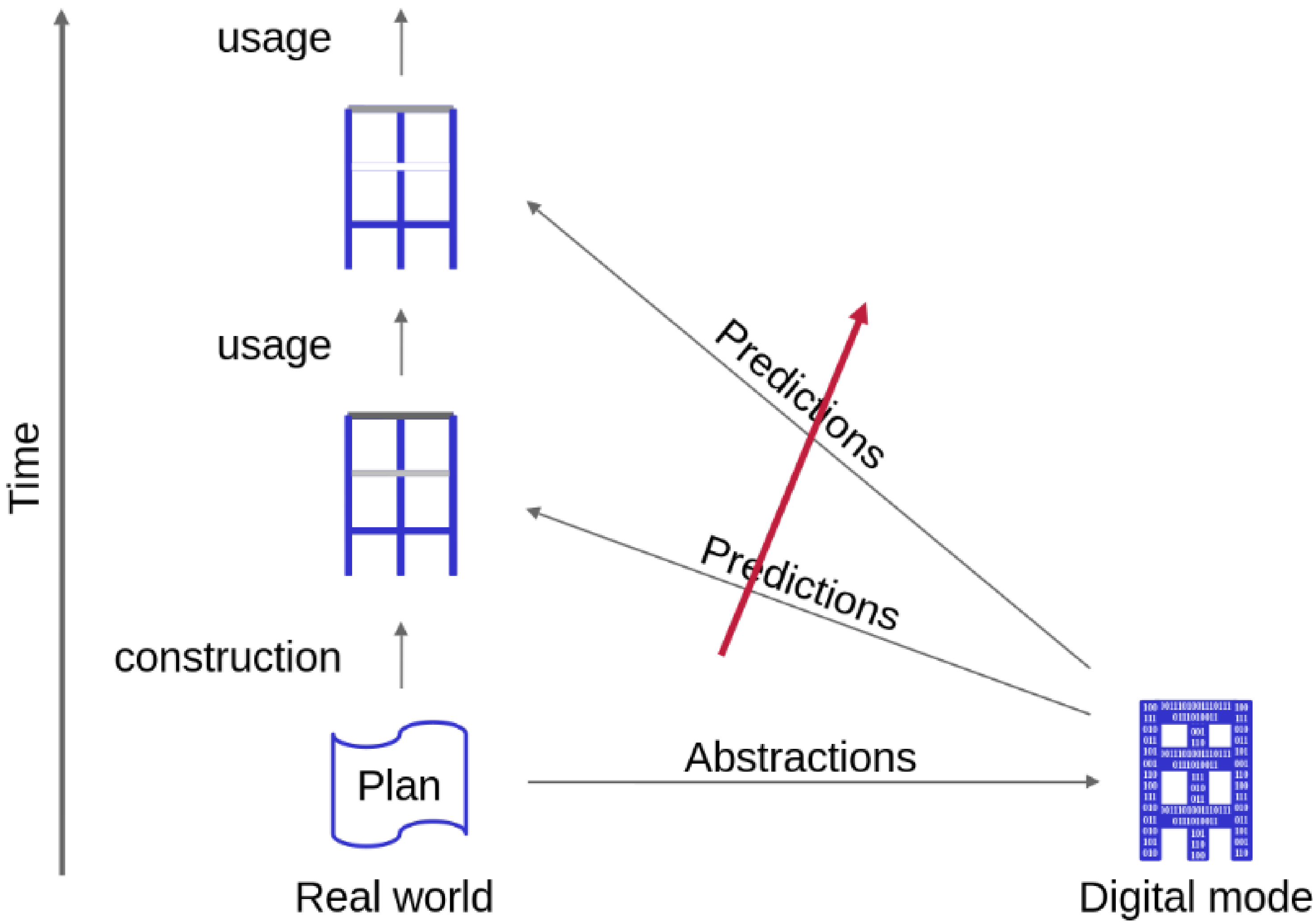


EXAMPLE: BRIDGE (4)

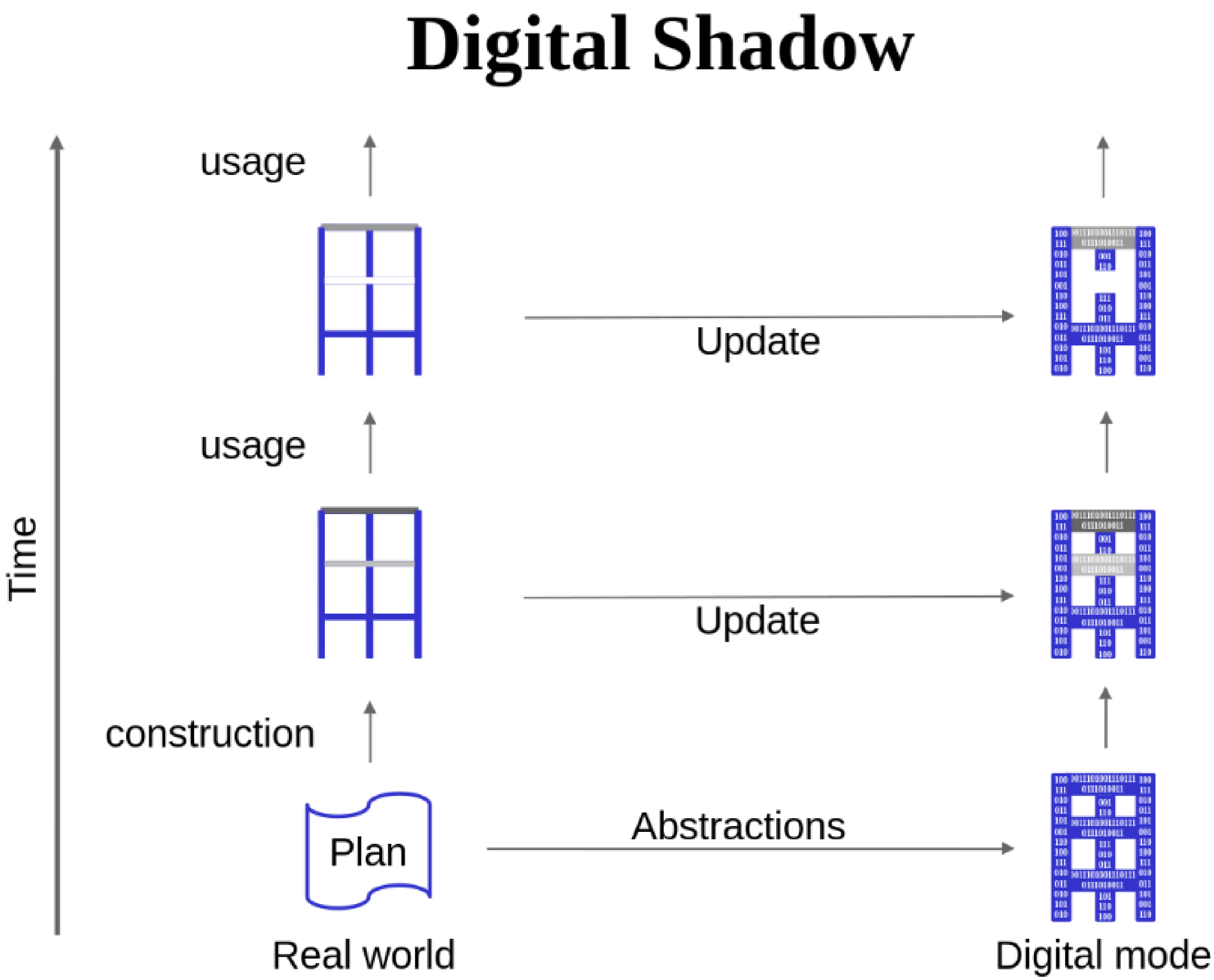


DEFINITIONS: DIGITAL MODEL

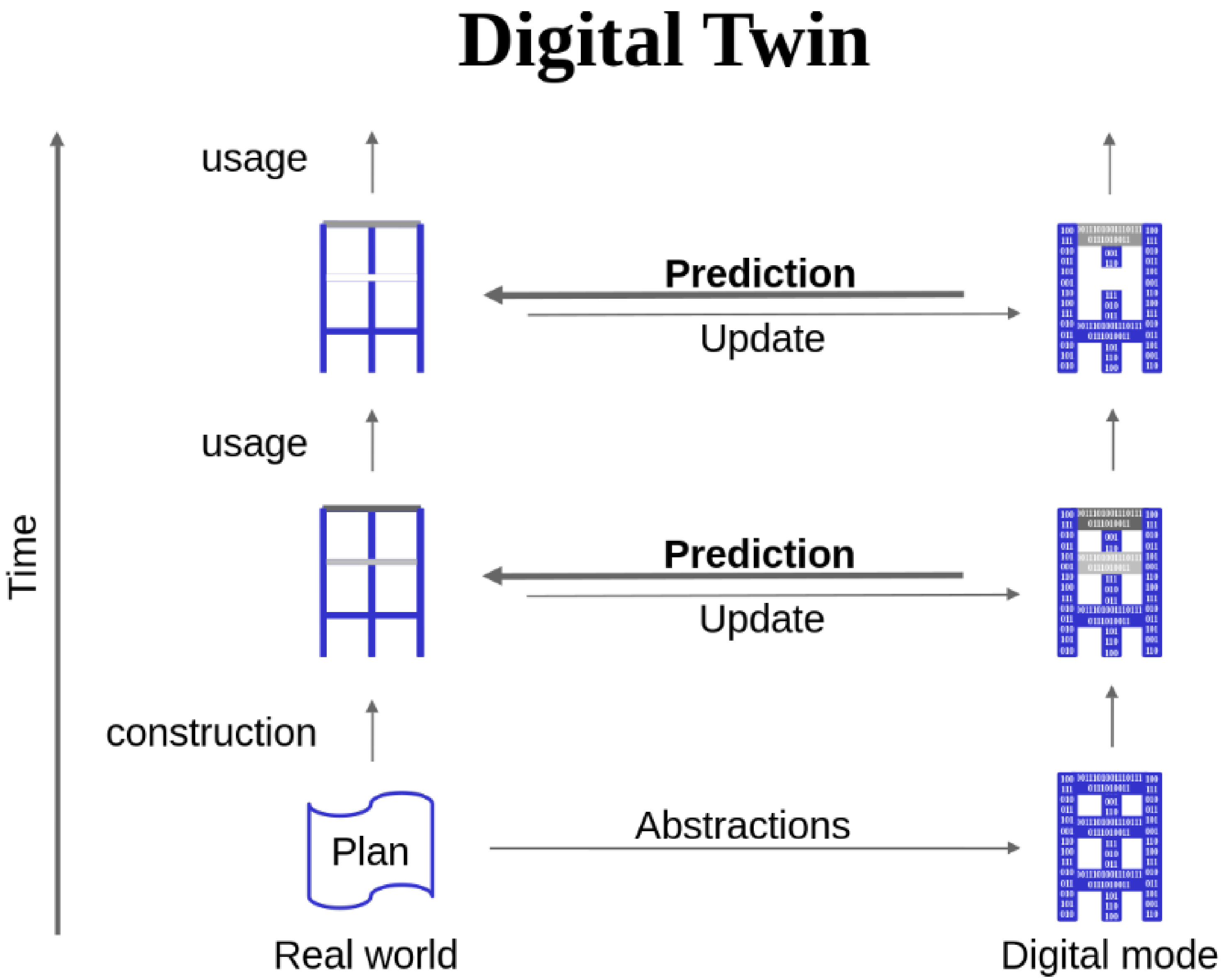
Digital Model



DEFINITIONS: DIGITAL SHADOW



DEFINITIONS: DIGITAL TWIN



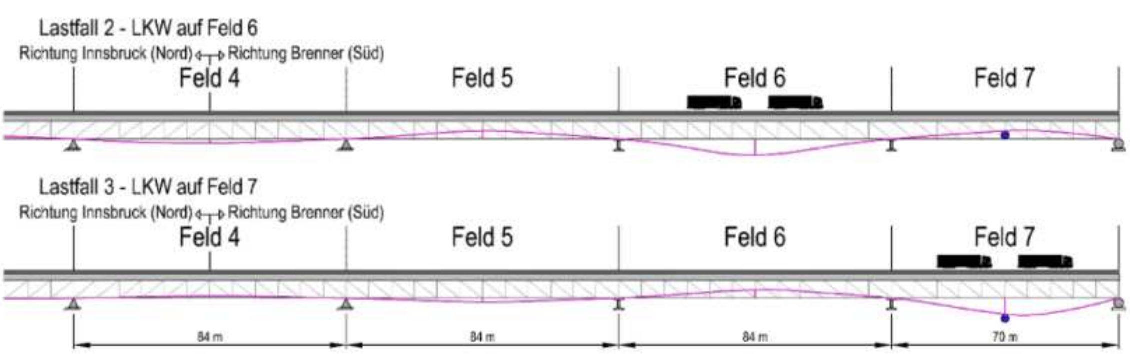
DIGITAL TWINS: Product of Megatrends

- Pervasive Use of CAD/Numerical Modeling
    - For Every Product/Process/Building/Patient/...
    $\rightarrow$ Have **Detailed Data** of ‘Real Thing’

- Pre-Compute, Only Then Build/Operate/Treat/...
    - Huge Reduction in Prototyping/Production Costs
    - Optimal Treatment of Patients
    $\rightarrow$ Have **Detailed Model(s)** of ‘Real Thing’

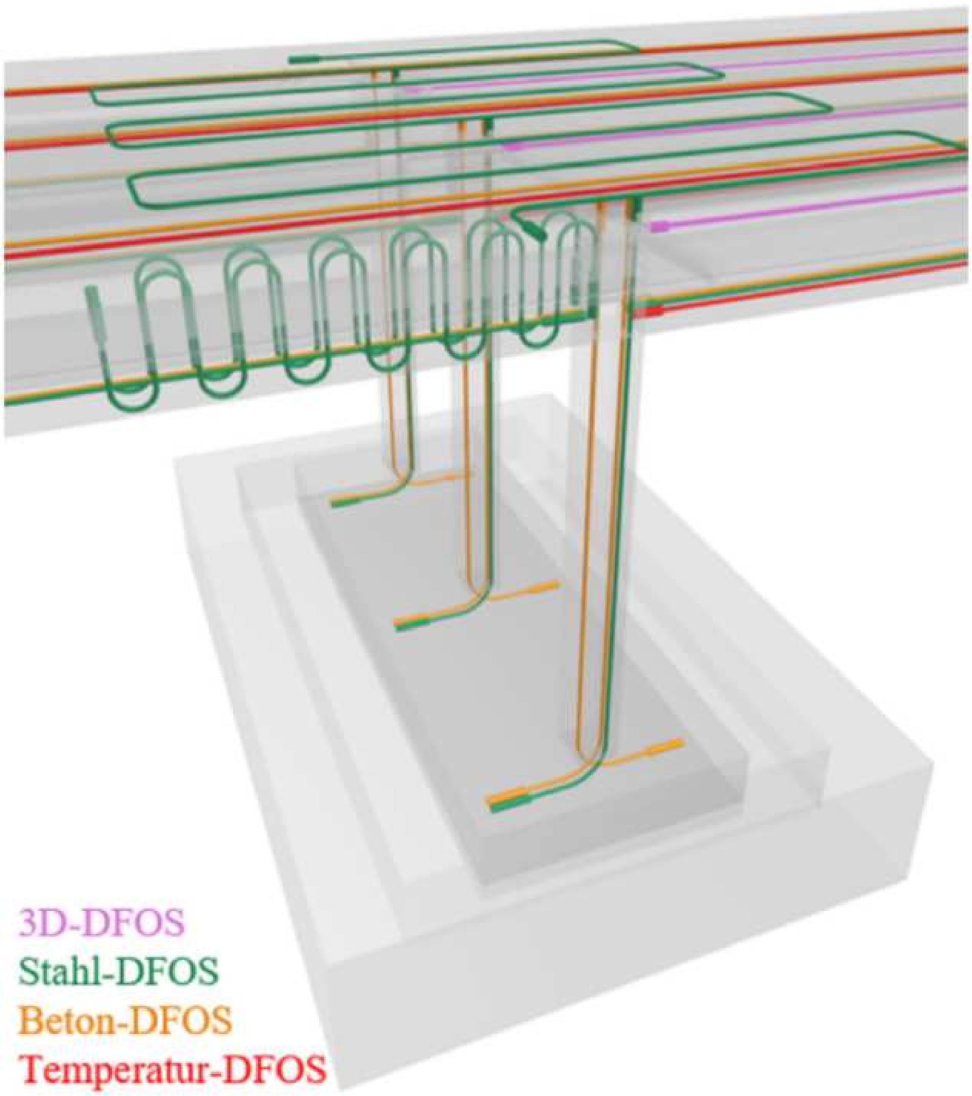
- Sensors Everywhere
    - Precise, Connected [G4,G5,...], Rugged, Cheap, ...
    $\rightarrow$ Can **Measure** the ‘Real Thing’

SENSORS FOR MONITORING BRIDGES



Courtesy: F. Schill, HS Mainz

FUTURE SENSORS FOR MONITORING CONCRETE



DIGITAL TWINS: Modus Operandi

- Have Digital Copy of Object/Process/Patient/...
- Equip Object/Process/Patient/... With Sensors
- With Data from Sensors (+Models):
  Infer State Normal, Weakening, Damaged, ...
- Update Digital Copy (+Models) Throughout Lifetime

DIGITAL TWINS: ‘GREAT EXPECTATIONS’

- Increased Safety
- Increased Comfort
- Longer Life Cycles (Assets, Processes, Humans)
- Optimal Process Control
- Reduced Environmental Footprint
- ...

DIGITAL TWINS: DATA LEVELS

- Description
- Enumeration of Parts
- Production/Replacement/Maintenance History
- Geometry for Display/CAD/Production
- Geometry/Abstraction for Modeling
    - Needs Proper Data for Each Discipline [CSD, CFD, CEM, CTD, ...]
    - Needs Proper Mesh/PDE Solver/... for Each Discipline
    - May Involve Extensive ‘De-Featuring
- Computational Mechanics Data
    - Material Data, Mesh, BC, Loads, ROMs,
- ...

DTs: WHAT LEVEL OF ABSTRACTION ?

- DT Is Not Reality, Only Model of Reality
    - Not Computing Each Atom All The Time $\Rightarrow$
- Need Abstraction Levels
- Partial/Ordinary Differential Equation(s)
    - CFD: Lifting Line/Potential/Euler/RANS/ LES/DNS
    - CSD: Lumped/Beam/Plate/Shell/Solid/
    - ...
- Numerics
    - FEM/FDM/FVM/...
- Model Abstraction from Numerics: ROMs/Surrogate Models
    - Modal, POD, PGD,...

DTs: CONSEQUENCES OF ABSTRACTION

- Level of Abstraction Determines:
    - Digital Twin
    - Data Needed for DT [CAD $\rightarrow$ DT]
    - Specialized Personnel Needed to Build DT
    - Software That Allows Seamless Updates
- Sensors
    - Type
    - Frequency of Measurements
    - Edge Computing

$\rightarrow$ **Determines Possible Type of Monitoring**

DTs: DELUGE OF DATA

- DTs for Every Object/Process/Patient/
- Constant Sensor Data
- Constant Update of DTs
- $\rightarrow$ Deluge of Data
- Who ?
    - Stores
    - Secures/Insures
    - Manages
    - Retrieves/Compares/Updates
    - Curates
    - ...

DTs OF ALL OF US

- Every Click, Every Web Search, Every Call
- Build Digital Twin of Human Behaviour/Thoughts
- Exploit Economically to the Maximum Extent
    - Directly: Advertising
    - Indirectly: 'Time on Subject'
- Examples:
    - Online Merchants: Amazon, Walmart, ...
    - Web Search Engines
    - Social Media: Facebook, Instagram, TikTok, ...
    - Hardware/Software: Apple, Microsoft, ...

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  - Hardware/Software: Apple, Microsoft, ...
- ‘We Know More About You Than You
- ‘Surveillance Capitalism [1789, 1989, ...]

LINEAR ELASTICITY

Define:

- ρ: Density
- $\mathbf{u}$: Displacement

$$\mathbf{u} = (u_x, u_y, u_z)$$

- σ: Stress Tensor

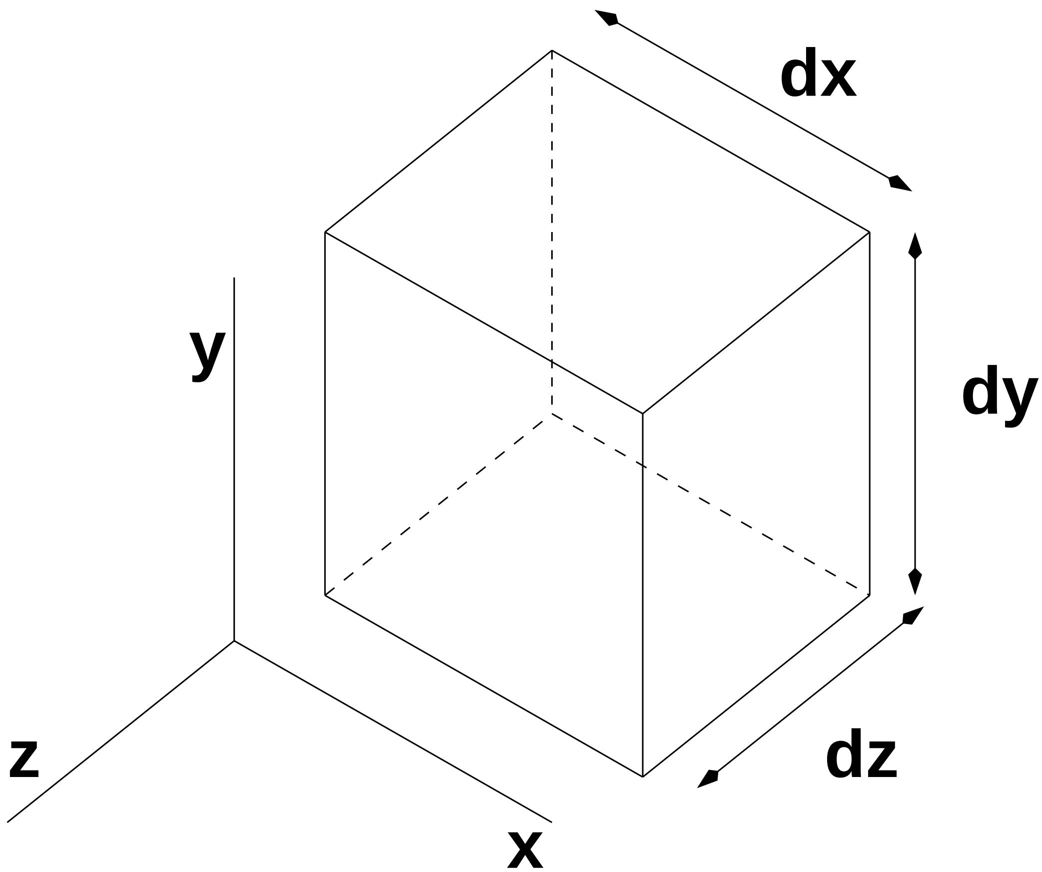
$$\sigma = \begin{bmatrix} \sigma^{xx} & \sigma^{xy} & \sigma^{xz} \\ \sigma^{yx} & \sigma^{yy} & \sigma^{yz} \\ \sigma^{zx} & \sigma^{zy} & \sigma^{zz} \end{bmatrix}$$

- ϵ: Strain Tensor

$$\epsilon = \frac{1}{2}\left[\nabla \mathbf{u} +^t \nabla \mathbf{u}\right]$$

$$\epsilon = \frac{1}{2}\begin{bmatrix} u_{x,x} + u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & u_{y,y} + u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & u_{z,z} + u_{z,z} \end{bmatrix}$$

CONSERVATION OF MOMENTUM (1)



CONSERVATION OF MOMENTUM (2)

Forces: x-Direction

- $x - direction$: x-Stress x Area at x:
  $dy\ dz\ \sigma^{xx}$
- $x - direction$: x-Stress x Area at $x + dx$:
  $dy\ dz\ [\sigma^{xx} + (\sigma^{xx})_{,x} dx + O(dx^2)]$
- $y - direction$: x-Shear x Area at y:
  $dx\ dz\ \sigma^{yx}$
- $y - direction$: x-Shear x Area at $y + dy$:
  $dx\ dz\ [\sigma^{yx} + (\sigma^{yx})_{,y} dy + O(dy^2)]$
- $z - direction$: x-Shear x Area at z:
  $dx\ dy\ \sigma^{zx}$
- $z - direction$: x-Shear x Area at $z + dz$:
  $dx\ dy\ [\sigma^{zx} + (\sigma^{zx})_{,z} dz + O(dz^2)]$

External Forces: f_x

Momentum Increment (Acceleration):

- $dx\ dy\ dz \rho\ \frac{d^2 u_x}{dt^2}$

CONSERVATION OF MOMENTUM (3)

Balance (x):
Momentum Increment = Sum of Forces $\Rightarrow$

$$dx\ dy\ dz\rho\ \frac{d^2 u_x}{dt^2} = dy\ dz(\sigma^{xx})_{,x} dx + dx\ dz(\sigma^{yx})_{,y} dy$$

$$+dx\ dy(\sigma^{zx})_{,z} dz + f_x + O(dx^4)$$

$\Rightarrow$ for $dx, dy, dy \to 0$:

$$\rho\ \frac{d^2 u_x}{dt^2} = \sigma^{xx}_{,x} + \sigma^{yx}_{,y} + \sigma^{zx}_{,z} + f_x$$

Same for y, z

$\Rightarrow$ Newton's Law

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \sigma + \mathbf{f} \qquad (E.1)$$

STRESS-STRAIN RELATIONSHIP

In General

$$\sigma = \mathbf{C}\epsilon$$

For Isotropic Material

$$\sigma = 2\mu\epsilon + \lambda tr(\epsilon)\mathbf{I}$$

$$\sigma^{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

- μ, λ: Lamé Parameters

Usual Material Parameters:

- Young's Modulus:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

- Poisson's Ratio:

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

STRESS-STRAIN RELATIONSHIP

Re-Write Constitutive Eqn. For Strains

$$\epsilon = \frac{1}{2\mu}\left[\sigma - \frac{\lambda}{2\mu + 3\lambda} tr(\sigma)\mathbf{I}\right]$$

$$\epsilon_{ij} = \frac{1}{2\mu}\left[\sigma^{ij} - \frac{\lambda}{2\mu + 3\lambda}\delta_{ij}\sigma^{kk}\right]$$

Special Case: Uniaxial Stress

- $\sigma^{11} = T$, All Other $\sigma^{ij} = 0$

$\Rightarrow$

$$\epsilon_{11} = \frac{1}{2\mu}\left[\sigma^{11} - \frac{\lambda}{2\mu + 3\lambda}\sigma^{11}\right]$$

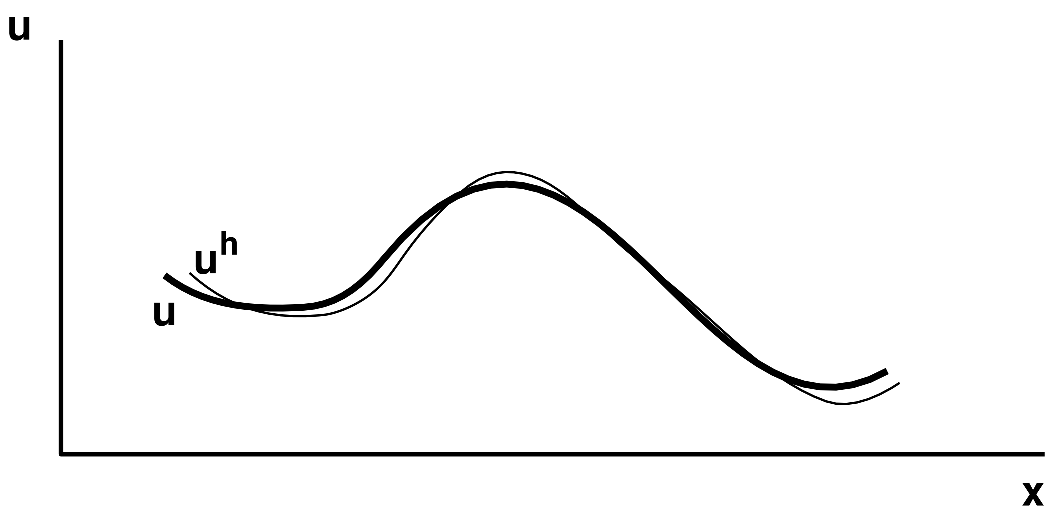
$\Rightarrow$ (Hooke's Law)

$$\sigma^{11} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\epsilon_{11}$$

APPROXIMATION THEORY (1)

To do: given $u(x)$ in Ω, approximate by known functions

$$u(x) \approx u^h(x) = f^i(x) a_i = N^i(x) \hat{u}_i$$



Approximation of Functions

APPROXIMATION THEORY (2)

Examples:

- Truncated Taylor Series

$$u(x) \approx u^h(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$$

$$a_j = \frac{1}{j!}\left.\frac{d^j u}{dx^j}\right|_{x=0}$$

- Truncated Sine Series

$$u(x) \approx u^h(x) = a_j sin\frac{j\pi x}{L}$$

$$a_j = \frac{2}{L}\int_0^L u(x) sin\frac{j\pi x}{L}dx$$

- Legendre Polynomials
- Hermite Polynomials

In General: Choose complete set of trial functions N^j:

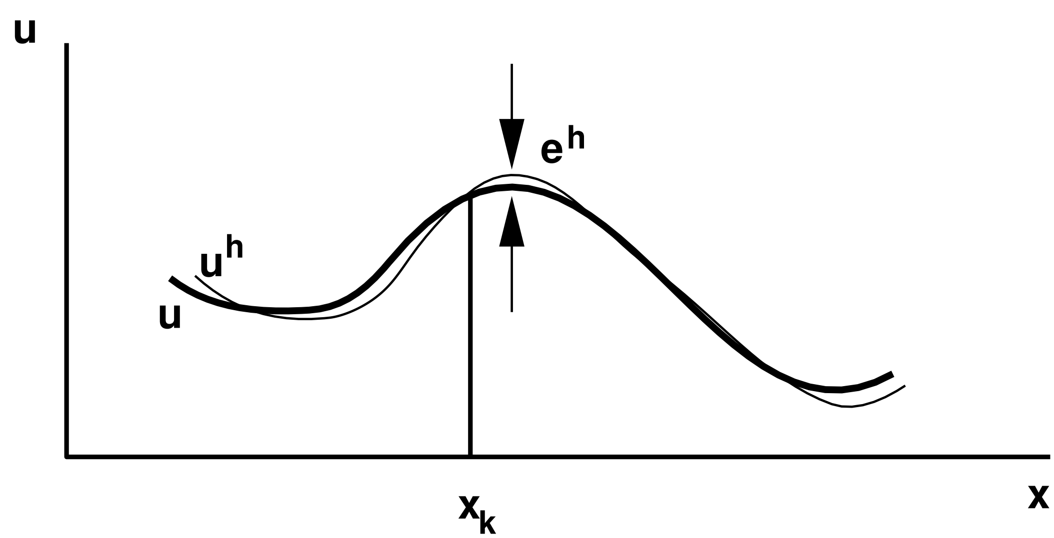
$$u(x) \approx u^h(x) = N^j a_j \quad ; \quad j = 1, 2, ...M$$

DETERMINATION OF CONSTANTS

1. Point Fitting: set $u^h = u$ at M selected points

$$u^h\big|_{x_k} \;=\; u\big|_{x_k} \quad k = 1, 2, ...M$$

$\Rightarrow$



Point Fitting

2. Weighted Residual Methods(WRM):

Define: $\epsilon^h = u - u^h$ - the error or residual

Require: $\epsilon^h \to 0$ in Ω

Introduce a set of weighting functions W^i ; $i = 1, 2, ...M$
Require that:

$$\int_\Omega W^i \epsilon^h d\Omega = 0 \quad , \quad i = 1, 2, ...M$$

Then, as $M \to \infty$, $\epsilon^h \to 0$ at all points in Ω

Insert expression for u^h:

$$\int_\Omega W^i \left(u - N^j a_j\right) d\Omega = 0$$

$\Rightarrow$

$$K^{ij} = \int_\Omega W^i N^j d\Omega \quad , \quad r^i = \int_\Omega W^i u d\Omega$$

Choice of W^i defines method !

2.1 Point Collocation

Choose: $W^i = \delta(x - x_i) \quad , \quad x_i \in \Omega$

WR statement

$$\int_\Omega \delta(x - x_i)\epsilon^h d\Omega = \epsilon^h(x_i) = 0 \quad , \quad i = 1, 2, ...M$$

$\Rightarrow$

$$N^j(x_i)a_j = u(x_i)$$

$\Rightarrow$ same as Point Fitting

2.2 Galerkin Method

Choose: $W^i = N^i$

WR statement

$$\int_\Omega N^i \epsilon^h d\Omega = \int_\Omega N^i \left(u - N^j a_j\right) d\Omega = 0 \quad , \quad i = 1, 2, ...M$$

$$\left[\int_\Omega N^i N^j d\Omega\right] a_j = \int_\Omega N^i u d\Omega \quad , \quad i = 1, 2, ...M$$

i.e.

Remarks:

LEAST SQUARES PROBLEM

$$I_{ls} = \int_\Omega (\epsilon^h)^2 d\Omega = \int_\Omega (u^h - u)^2 d\Omega$$

$$= \int_\Omega (N^k a_k - u)^2 d\Omega \to min$$

$\Rightarrow$

$$\delta I_{ls} = \delta a_k \int_\Omega N^k (N^l a_l - u) d\Omega = 0$$

$$\int_\Omega N^k N^l d\Omega \; a_l = \int_\Omega N^k u d\Omega$$

$\Rightarrow$ Equivalent to Galerkin WRM

$\Rightarrow$ Choice of W^i from same set as N^i optimal

DRAWBACKS OF GLOBAL TRIAL FUNCTIONS

1. Determining N^j's difficult for all but the simplest geometries in 2/3D

problems (can use strongly orthogonal polynomials)

4. a_j's have no physical significance

LOCAL TRIAL FUNCTIONS

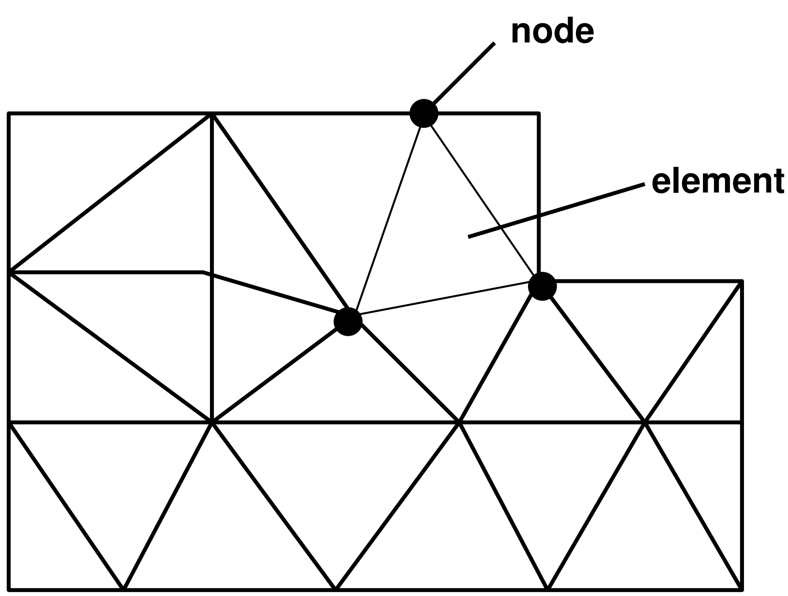
Given $u(x)$, $x \in \Omega$:

Divide Ω into a set of non-overlapping sub-intervals Ω_{el}

Define u^h in each sub-interval

Sub-intervals - ELEMENTS

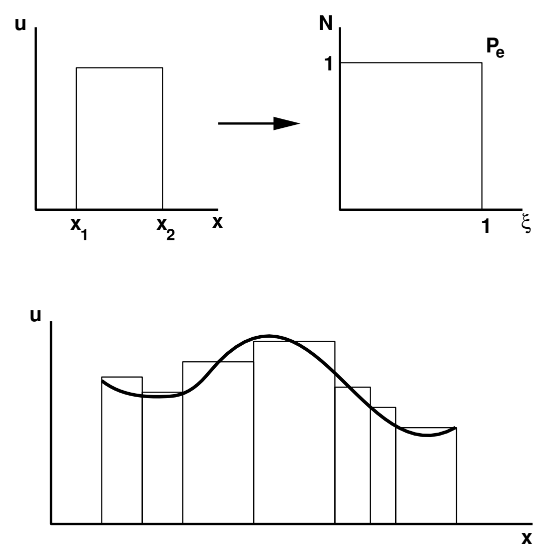
$x_1, x_2, ...$ - NODES



CONSTANT TRIAL FUNCTIONS

Define a piecewise constant function

$$P^E = \begin{cases} 1 & \text{in element } E \\ 0 & \text{in all other elements} \end{cases}$$



Constant Trial Functions

Then globally

$$u \approx u^h = P^E u_E$$

Locally, on element el,

$$u \approx u^h = u_{el}$$

LINEAR TRIAL FUNCTIONS

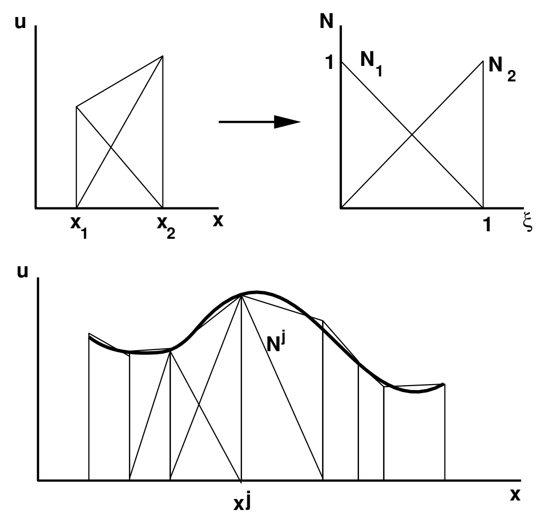
Better approximation: let u^h vary linearly

Place nodes at end of each element

Define a piecewise linear trial function

$$N^j = \begin{cases} 1 & \text{at node } j \\ 0 & \text{at all other nodes} \end{cases}$$

and N^j non-zero only on elements associated with node j



Linear Trial Functions

Globally

$$u \approx u^h = N^j(x)u(x_j) = N^j(x)\hat{u}_j$$

Locally over element el with nodes 1 and 2

$$u \approx u^h = N^1\hat{u}_1 + N^2\hat{u}_2$$

$\xi = (x - x_1)/(x_2 - x_1)$

$N^1_{el} = (x_2 - x)/h_{el} = 1 - \xi$

$N^2_{el} = (x - x_1)/h_{el} = \xi$

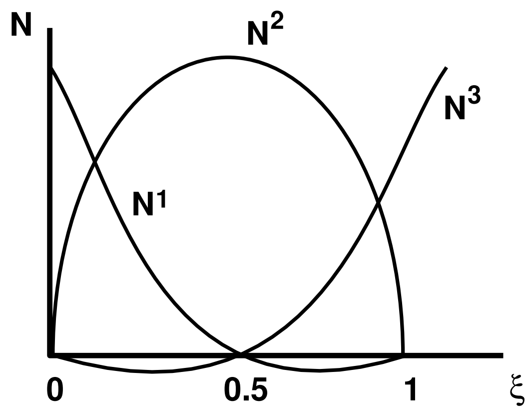
$u^h = \frac{(x_2-x)u_1+(x-x_1)u_2}{h_{el}} = (1-\xi)u_1 + \xi u_2$

Observe that: $x = (1-\xi)x_1 + \xi x_2 = N^1 x_1 + N^2 x_2$

QUADRATIC TRIAL FUNCTIONS

Better approximation: let u^h vary quadratically

Place nodes at end of each element, as well as the middle



Quadratic Trial Functions

$$N^1_{el} = (1-\xi)(1-2\xi)$$

$$N^2_{el} = 4\xi(1-\xi)$$

$$N^3_{el} = -\xi(1-2\xi)$$

GENERAL PROPERTIES OF SHAPE-FUNCTIONS

1. Interpolation Property:

$$u^h = N^i(x)\hat{u}_i$$

$\Rightarrow$

$$u^h(x_j) = N^i(x_j)\hat{u}_i = \hat{u}_j \Rightarrow N^i(x_j) = \delta^i_j$$

2. Constant Sum: Must be able to represent a constant

$$u = 1 \;\Rightarrow\; u^h = 1 = N^i(x)\hat{u}_i$$

but interpolation property $\Rightarrow\; \hat{u}_i = 1 \;\Rightarrow$

$$\sum_i N^i(x) = 1 \quad , \quad \forall x \in \Omega \qquad (*)$$

3. Conservation property: from Eqn.$(*)$:

$$\sum_i N^i_{,k} = 0 \quad , \quad \forall x \in \Omega_{el}$$

WRM OF APPROXIMATION WITH LOCAL FUNCTIONS

Basic Idea:

$$\int_{\Omega} ... = \sum_{el} \int_{\Omega_{el}} ... \qquad (*)$$

$\Rightarrow$

- build integrals on element level
- gather info from global point-arrays to local element-arrays
- scatter-add resulting integrands to global rhs/matrix locations

$$K^{ij} u_j = \left[\sum_{el} K^{ij}_{el} \right] [u_j]_{el} = \sum_{el} r^i_{el} = r^i$$

Note: for Eqn.($*$) only need info: nodes belonging to an element

$\Rightarrow$ drastic simplification of data structures/logic

FROM APPROXIMATION TO OPERATORS

Before: given u, approximate: $\|u - u^h\| \rightarrow min$

Now: given $L(u) = 0$ approximate: $\|L(u) - L(u^h)\| \rightarrow min$

$\Rightarrow \qquad \|L(u^h)\| \rightarrow min$

Minimize error:

$$\epsilon_L^h = L(u^h) = L(N^i \hat{u}_i)$$

using WRM

$$\int_\Omega W^i \epsilon_L^h d\Omega = 0 \quad , \quad i = 1, M$$

Choice of N^i, W^i defines the method:

- N^i polynomial, $W^i = \delta(x_i)$: FDM
- N^i polynomial, $W^i = 1$ if x$\subset \Omega_{el}$, 0 otherwise : FVM
- N^i polynomial, $W^i = N^i$: GFEM
- N^i polynomial, $W^i \neq N^i$: Petrov-GFEM
- N^i spectral, $W^i = \delta(x_i)$: SEM
- N^i, W^i NURBS: Isogeometric G/PG

POISSON OPERATOR

Given

$$\nabla^2 u = f \quad in \quad \Omega \quad , \quad u = 0 \quad on \quad \Gamma$$

WRM

$$\int_\Omega W^i \ \nabla^2 N^j \ d\Omega \ \hat{u}_j = \int_\Omega W^i \ f \ d\Omega$$

$\Rightarrow$

- N^j should have defined 2nd order derivatives
  $\Rightarrow C^1$-continuous across elements
- W^i can be the δ-function

Integration by parts:

$$-\int_\Omega \nabla W^i \cdot \nabla N^j \ d\Omega \ \hat{u}_j = \int_\Omega W^i \ f \ d\Omega$$

$\Rightarrow$

- Order of max(derivative) reduced $\Rightarrow$ can use wider space of trial/weight functions
- N^j should have defined 1st order derivatives
  $\Rightarrow C^0$-continuous across elements
- W^i can not be the δ-function

MINIMIZATION PROBLEM

$$I_{rz} = \int_\Omega [\nabla \epsilon^h]^2 \; d\Omega = \int_\Omega [\nabla (u^h - u)]^2 \; d\Omega \rightarrow min$$

$\Rightarrow$

$$\delta I_{rz} = \delta \hat{u}_i \int_\Omega \nabla N^i \cdot (\nabla N^j \hat{u}_j - \nabla u) \; d\Omega = 0$$

but

$$-\int_\Omega \nabla N^i \cdot \nabla u \; d\Omega = \int_\Omega N^i \nabla^2 u \; d\Omega = \int_\Omega N^i f \; d\Omega$$

$\Rightarrow$

$$\delta I_{rz} = \delta \hat{u}_i \left[\int_\Omega \nabla N^i \cdot \nabla N^j \; d\Omega \; \hat{u}_j + \int_\Omega N^i \; f \; d\Omega \right] = 0$$

$\Rightarrow$ equivalent to Galerkin WRM

$\Rightarrow$ choice of W^i from same set as N^i optimal

1D ELASTICITY (1)

$$-\frac{\partial}{\partial x} E \frac{\partial u}{\partial x} = q$$

Approximate: $u = N^i \hat{u}_i$

<u>WRM+Integration by parts+Galerkin:</u>

$$\int_\Omega \frac{\partial N^i}{\partial x} E \frac{\partial N^j}{\partial x} \; dx \; \hat{u}_j = \int_\Omega N^i N^j \; dx \; \hat{q}_j$$

1D ELASTICITY (2)

For Linear Shape Functions:

$$\mathbf{K}_{el} = \int_{\Omega} \frac{\partial N^i}{\partial x} E \frac{\partial N^j}{\partial x} \, dx = \frac{E}{\Delta x} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{M}_{el} = \int_{\Omega} N^i N^j \, dx = \frac{\Delta x}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{M}_{el}^{l} = \frac{\Delta x}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

REFERENCES

[1] O.C. Zienkiewicz and K. Morgan - *Finite Elements and Approximation*; J. Wiley & Sons (1983).

MOTIVATION (1)

- All Structures Age

- 1st World: Concrete [Bridges, Buildings, ...]
    - Life Span: 60-80 Years (Weathering, Cracks, ...)
    - Many Bridges and Buildings Nearing That Age

$\rightarrow$ **Infrastructure Crisis**

- Same for Many Other Applications
    - Turbines
    - Composites [Wings, Propellers, ...]
    - Objects Subjected to Harsh Environments

RECENT BRIDGE FAILURES

RECENT BRIDGE FAILURES

MOTIVATION (2)

As All Structures Age:

Q1: Given Loads and Measurements:
Can We Infer State of Material (Weakening) ?

Q2: If We Know/Suspect Weakening:
Where Should We Reinforce ?

MOTIVATION (3)

- Problem of High Importance
  $\Rightarrow$ Considerable Body of Work
- Frequency Domain (Sensors: Accelerometers)
    - Since Mid-70s
    - Large Effort at Sandia National Labs
    - Easy to Detect **That**, But Not **Where**
- Time Domain
    - Several (Some Adjoints in Time-Domain)
- Steady: Adjust/Approximate FEM Model from Measurements
    - Ladeveze et al.
    - Aubry et al. (Adjoints+Patches)
- New Here: DT, Continuous Monitoring, High-Fidelity

ASSUMPTIONS (1)

- Monitoring Via Loads and Measurements:
  - n (Standard) Loads $\mathbf{f}^i, i = 1, n$ Given
  - n Displacements/Strains Measured at m Locations
- Weakening Can Occur at Any Location
  - Most Conservative
- Weakening Described by Field $0 < \alpha(\mathbf{x}) < 1$
- Deformations and Strains Well Described by FEM:

$$\mathbf{K} \cdot \mathbf{u}_i = \mathbf{f}_i \quad , \quad i = 1, n \quad , \quad \mathbf{K} = \sum_{el} \alpha_{el} \mathbf{K}_{el}$$

ASSUMPTIONS (2)

- Sensors Limited by Signal/Noise Ratios

$$|\mathbf{u}^m| > u_0 \quad , \quad |\mathbf{s}^m| > s_0$$

- Forces Used to Monitor Structure Limited by Practical Considerations
  $\Rightarrow$ Not Arbitrary

LIMITATIONS (1)

Simple Truss, Clamped at $x = 0$, Force f_c Applied at $x = L$

Elasticity Equations Reduce to:

$$f = f_c \quad , \quad E\frac{du}{dx} = f_c$$

Integrate:

$$\frac{du}{dx} = \frac{f_c}{E}$$

$\Rightarrow$

$$u(L) - u(0) = f_c \int_0^L \frac{1}{E} dx$$

Or:

$$\int_0^L \frac{1}{E} dx = \frac{u(L) - u(0)}{f_c}$$

LIMITATIONS (2)

Suppose:

- f_c: Known (Measured)
- $u(L)$: Known (Measured)

Q: Can We Determine $E(x)$?

A: **Only If** $E(x) = E_c$ **(i.e. Constant)** !

A: **Not If** $E(x)$ **Arbitrary**

Consider n Regions $\Delta x_i, i = 1, n$; E_i Constant in Each Region

$\Rightarrow$

$$\sum_i \frac{\Delta x_i}{E_i} = \frac{u(L) - u(0)}{f_c}$$

$\Rightarrow$ Infinitely Many Possible Solutions

LIMITATIONS (3)

Simple Case:

- $L = 1, u(0) = 0, u(1) = 1, f_c = 1$
- 4 Equal Regions of $\Delta x = 0.25$

$\Rightarrow$

$$\sum_{i=1}^{4} \frac{1}{k_i} = 4$$

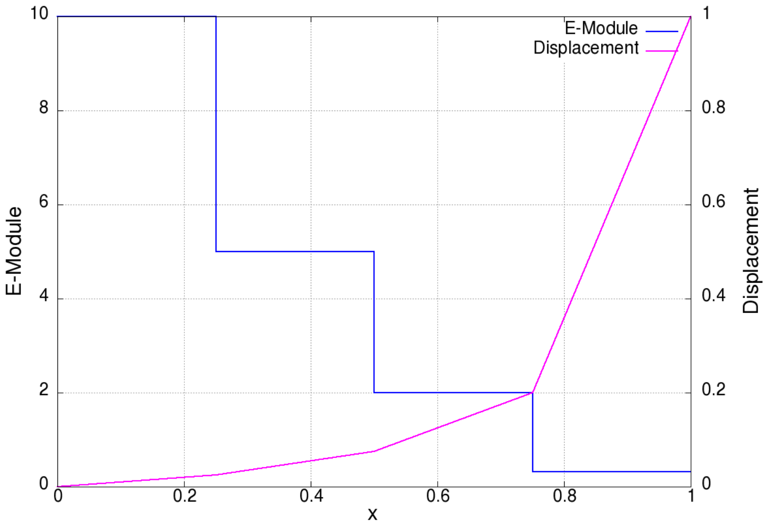
Options:

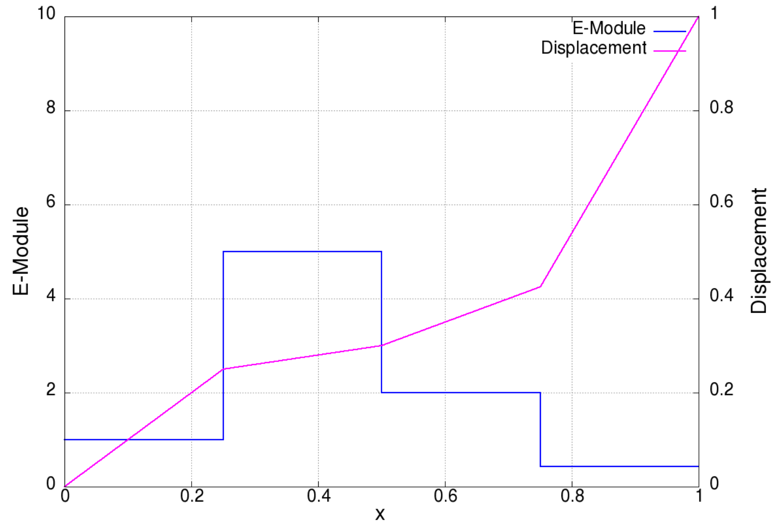
a) $E_1 = E_2 = E_3 = E_4 = 1;$
b) $E_1 = E_2 = 2; E_3 = E_4 = 2/3;$
c) $E_1 = 10, E_2 = 5, E_3 = 2, E_4 = 10/32;$
d) ...

Moreover: Spatial Sequence of $E's$ Can be Changed Without Affecting the Resulting $u(L)$

$\Rightarrow$ Highly Oscillatory Distributions of E Admissible

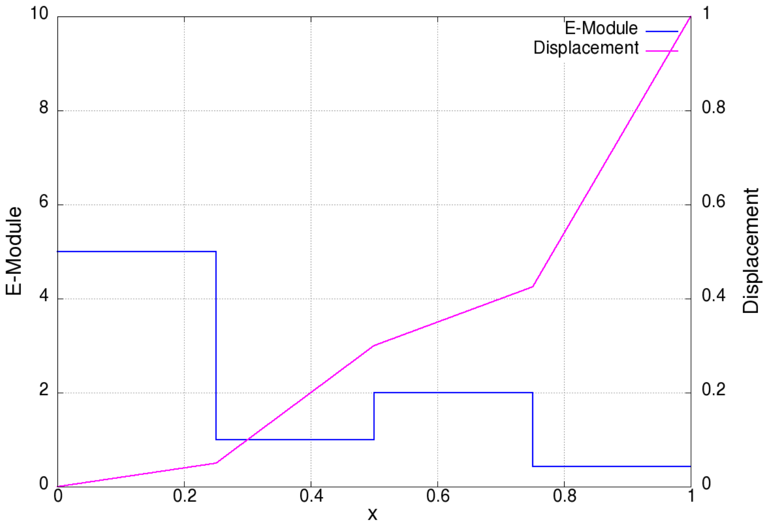
LIMITATIONS (4)

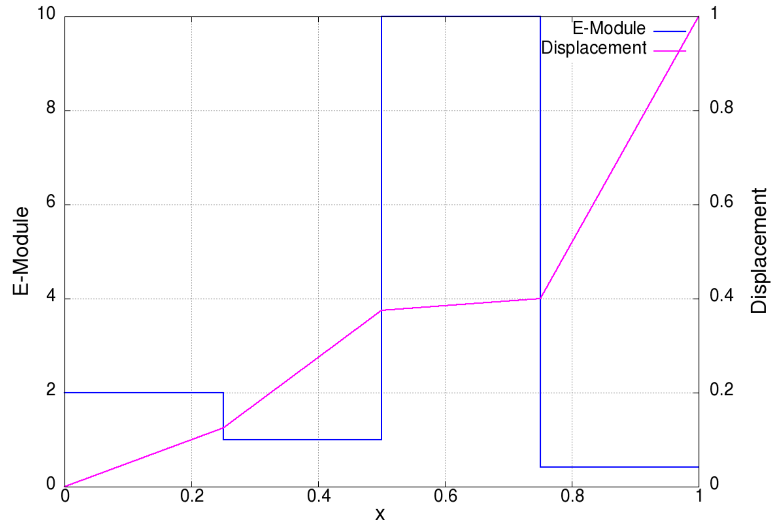




1D Case: Possible E-Modules and Resulting Deformations for $f_c = 1$

LIMITATIONS (5)





1D Case: Possible E-Modules and Resulting Deformations for $f_c = 1$

FORMULATE AS OPTIMIZATION PROBLEM

Find $\alpha(\mathbf{x})$ Such That

$$I(\mathbf{u}_{1,..,n}, \alpha) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}^{md} (\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)^2$$

$$+\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}^{ms} (\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s} \mathbf{s}_i)^2 \rightarrow min$$

Subject To:

$$\mathbf{K} \cdot \mathbf{u}_i = \mathbf{f}_i \quad , \quad i = 1, n \quad , \quad \mathbf{K} = \sum_{el} \alpha_{el} \mathbf{K}_{el}$$

EXTEND TO LAGRANGIAN FUNCTIONAL

$$L(\mathbf{u}_{1,..,n}, \alpha, \tilde{\mathbf{u}}_{1,..,n}) = I(\mathbf{u}_{1,..,n}, \alpha) + \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t (\mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i)$$

- $\tilde{\mathbf{u}}$: Adjoint Variables

Variation of the Lagrangian WRT Each Component

$$\frac{dL}{d\tilde{\mathbf{u}}_i} = \mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i = 0$$

$$\frac{dL}{d\mathbf{u}_i} = \sum_{j=1}^{m} w_{ij}^{md} \mathbf{I}_{ij}^{d} \cdot (\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)$$

$$+ \sum_{j=1}^{m} w_{ij}^{ms} \mathbf{J}_{ij}^{s} (\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s} \mathbf{s}_i) + \mathbf{K}^t \tilde{\mathbf{u}}_i = 0$$

$$\frac{dL}{d\alpha_e} = \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \cdot \frac{d\mathbf{K}}{d\alpha_e} \cdot \mathbf{u}_i = \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \cdot \mathbf{K}^e \cdot \mathbf{u}_i \quad ,$$

- $\mathbf{J}_{ij}^s$: Relationship Displacements/Strains

USE OF ADJOINT SOLVERS

- Consequences
  - Gradient of L wrt α: n Forward/Adjoint Solves;
  - **Independent** of Number of Variables of α (!);
  - Once Forward/Adjoint Solves Done: Cost of Gradient of Each α_{el} Is $O(1)$;

  $\Rightarrow$ Can Use Detailed FEM Models $\rightarrow$ Detailed Digital Twin
  - 'Hi-Fi Digital Twin'
  - Based on Algebraic Equations (FEMs), Not PDEs $\Rightarrow$ General

- Most Structural Problems: $\mathbf{K} =^t \mathbf{K} \Rightarrow$
  - Direct Solvers: Cost of Adjoint Negligible
  - Iterative Solver: Preconditioner Can Be Re-Utilized

OPTIMIZATION CYCLE

- For Each Force/Measurement Pair i:
- With Current α: Obtain Displacements $\rightarrow \mathbf{u}_i$
- With Current α, $\mathbf{u}_i$, $\mathbf{u}_{ij}^{md}, \mathbf{s}_{ij}^{md}$: Obtain Adjoints $\rightarrow \tilde{\mathbf{u}}_i$
- With $\mathbf{u}_i, \tilde{\mathbf{u}}_i$: Obtain Gradients $\rightarrow I^i_{,\alpha} = L^i_{,\alpha}$
- Sum up the gradients $\rightarrow I_{,\alpha} = \sum_{i=1}^n I^i_{,\alpha}$
- If Necessary: Smooth Gradients $\rightarrow I^{smoo}_{,\alpha}$
- Update $\alpha_{new} = \alpha_{old} - \gamma I^{smoo}_{,\alpha}$.
- γ: Stepsize

INTERPOLATION OF DISPLACEMENTS AND STRAINS

Displacements:

$$\mathbf{u}_i(\mathbf{x}_i^m) = \mathbf{I}_i^d(\mathbf{x}_i^m) \cdot \mathbf{u}$$

Strains:

$$\mathbf{s} = \mathbf{D}\mathbf{u}$$

$$\mathbf{s}_i(\mathbf{x}_i^m) = \mathbf{I}_i^s(\mathbf{x}_i^m) \cdot \mathbf{s} = \mathbf{I}_i^s(\mathbf{x}_i^m) \cdot \mathbf{D} \cdot \mathbf{u}$$

CHOICE OF WEIGHTS (1)

Cost Function:

$$I(\mathbf{u}_n, \alpha) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{md}(\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{ms}(\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s}\mathbf{s}_i)^2$$

Problem: **Dimensionally Inconsistent** (!)

CHOICE OF WEIGHTS (2)

Local Weighting:

$$w_{ij}^{md} = \frac{1}{(\mathbf{u}_{ij}^{md})^2} \quad ; \quad w_{ij}^{ms} = \frac{1}{(\mathbf{s}_{ij}^{ms})^2}$$

- Works Well
- May Lead to ‘Over-Emphasis’ of Small Displacements/Strains (That May Be In Regions of Marginal Interest)

CHOICE OF WEIGHTS (3)

Average Weighting

$$u_{av} = \frac{\sum_{j=1}^{m} |\mathbf{u}_{ij}^{md}|}{m} \quad ; \quad w_{ij}^{md} = \frac{1}{u_{av}^2}$$

$$s_{av} = \frac{\sum_{j=1}^{m} |\mathbf{s}_{ij}^{ms}|}{m} \quad ; \quad w_{ij}^{ms} = \frac{1}{s_{av}^2}$$

- Works Well
- May Lead to 'Under-Emphasis' of Small Displacements/Strains
  (That May Be In Important Regions)

CHOICE OF WEIGHTS (4)

Max Weighting

$$u_{max} = max(|\mathbf{u}_{ij}^{md}|, j = 1, m) \quad ; \quad w_{ij}^{md} = \frac{1}{u_{max}^2}$$

$$s_{max} = max(|\mathbf{s}_{ij}^{ms}|, j = 1, m) \quad ; \quad w_{ij}^{ms} = \frac{1}{s_{max}^2}$$

- Works Well
- May Lead to 'Under-Emphasis' of Smaller Displacements/Strains (That May Be In Important Regions)

CHOICE OF WEIGHTS (5)

Local/Max Weighting

$$w_{ij}^{md} = \frac{1}{max(\epsilon u_{max}, |\mathbf{u}_{ij}^{md}|))^2}$$

$$w_{ij}^{ms} = \frac{1}{max(\epsilon s_{max}, |\mathbf{s}_{ij}^{ms}|))^2}$$

- $\epsilon = O(0.01 - 0.10)$
- Combines Local Weighting With Max-Bound Minimum for Local Values
- Seemed to Work Best

SMOOTHING OF GRADIENTS (1)

Gradient of α Oscillatory, ‘Lives in H^{-1}’

$\Rightarrow$ Need to Smooth, ‘Regularize’

Can Smooth α or $I_{,\alpha}$

Consider α

SMOOTHING OF GRADIENTS (2)

Option 1: Simple Element/Point/Element Averaging

Step 1: Element to Point

$$\alpha_p = \frac{\sum_e \alpha_e V_e}{\sum_e V_e}$$

Step 2: Point to Element

$$\alpha_e = \frac{1}{n_e} \sum_i \alpha_i$$

- α_e: Weakness in Element e
- V_e: Volume of Element e
- $\sum_e$: All Elements Surrounding Point p
- α_p: Weakness Averaged to Point p
- n_e: Nr. of Nodes/Points of Element e

- Simple, 'Crude'
- Works Surprisingly Well

SMOOTHING OF GRADIENTS (3)

Option 2: H^1 (Weak) Laplacian Smoothing

$$\left[1 - \lambda\nabla^2\right] \alpha = \alpha_0 \quad , \quad \alpha_{,n}\big|^{\Gamma} = 0$$

- α_0: Initial, Unsmoothed Values
- α: Smoothed Values
- λ: Free Parameter (Problem and Mesh Dependent

After Discretization:

$$\left[\mathbf{M}_c + \lambda \mathbf{K}_d\right] \boldsymbol{\alpha} = \mathbf{M}_{p1p0} \boldsymbol{\alpha}_0$$

- $\mathbf{M}_c$: Consistent Mass Matrix (L2 Matrix)
- $\mathbf{K}_d$: 'Diffusion Matrix' (Laplacian)
- $\mathbf{M}_{p1p0}$: Projection Matrix (Element $\rightarrow$ Points)

- Solved Iteratively (Fast Convergence)
- Problem: Choice of λ

SMOOTHING OF GRADIENTS (4)

Option 3: Pseudo-Laplacian Smoothing

$$\left[1 - \lambda \nabla h^2 \nabla\right] \alpha = \alpha_0$$

- h: Characteristic Element Size

After Discretization:

$$\left[\mathbf{M}_c + \lambda \mathbf{K}_{h^2}\right] \boldsymbol{\alpha} = \mathbf{M}_{p1p0} \boldsymbol{\alpha}_0$$

- $\mathbf{K}_{h^2}$: 'h^2-Scaled Diffusion Matrix' (h^2 Laplacian)

For Linear Elements (Simplex) Equivalent To

$$\left[\mathbf{M}_c + \lambda \left(\mathbf{M}_l - \mathbf{M}_c\right)\right] \boldsymbol{\alpha} = \mathbf{M}_{p1p0} \boldsymbol{\alpha}_0$$

- $\mathbf{M}_l$: Lumped Mass Matrix
- Typical Value: $\lambda = 0.05$

SMOOTHING OF GRADIENTS (5)

Option 4: Smoothing via Convolution

$$\alpha = \frac{\int_{\Omega_h} G(\mathbf{x})\alpha_0 d\Omega}{\int_{\Omega_h} G(\mathbf{x}) d\Omega}$$

- α_0: Initial, Unsmoothed Values
- α: Smoothed Values
- Ω_h: Local Domain
- $G(\mathbf{x})$: Kernel Function (e.g. Gaussian)

Obervations:

- Choice of Local Domain
  - 3-4 Elements ? Constant ?
- Similar to Pseudo-Laplacian Smoothing

SMALL EXAMPLE: 1-D TRUSS (1)

Define Case:

- 1-D
- E, A Constant
- $u(0) = 0$
- One Load Only at $x = L$: $f(L) = f_0$
- 2 Linear Elements
- Displacement Sensors at Each Point
- Weights for Displacements: $w_i = 1.0$

- Assume: Structure Weakened to $\alpha = 0.5$
- $\Rightarrow$ Exact Solution (Proper E, A, L):
  $u_0 = 0, u_1 = 1, u_2 = 2$
- $\Rightarrow$ Measurements:
  $u_0^m = 0, u_1^m = 1, u_2^m = 2$
- Then: Exact Solution for $\alpha = 1.0$
  $u_0 = 0, u_1 = 0.5, u_2 = 1$

SMALL EXAMPLE: 1-D TRUSS (2)

First Iteration: Start With $\alpha = 1.0$ (No Weakening)

$$\frac{dL}{d\tilde{\mathbf{u}}} = \mathbf{K} \cdot \mathbf{u} - \mathbf{f} = 0$$

$\Rightarrow$

$$u_0^{FEM} = 0, u_1^{FEM} = 0.5, u_2^{FEM} = 1$$

Adjoint:

$$\frac{dL}{d\mathbf{u}} = \sum_{j=1}^{m} w_j^{md} \mathbf{I}_j^d \cdot (\mathbf{u}_j^{md} - \mathbf{I}_j^d \cdot \mathbf{u}) + \mathbf{K}^t \tilde{\mathbf{u}} = 0$$

As Measuring Points Coincident With FEM Points:

$$\mathbf{I}_j^d(x_i) = \delta_{ij}$$

SMALL EXAMPLE: 1-D TRUSS (3)

$\Rightarrow$ RHS for System $\mathbf{K}^t\tilde{\mathbf{u}} = \mathbf{r}$

- $r_0 = u_0^m - u_0^{FEM} = 0.0$
- $r_1 = u_1^m - u_1^{FEM} = 0.5$
- $r_2 = u_2^m - u_2^{FEM} = 1.0$

$\Rightarrow \tilde{u}_0 = 0, \tilde{u}_1 > 0, \tilde{u}_2 > \tilde{u}_1 > 0$

$$\frac{dL}{d\alpha_e} = \tilde{\mathbf{u}}^t \cdot \frac{d\mathbf{K}}{d\alpha_e} \cdot \mathbf{u} = \tilde{\mathbf{u}}^t \cdot \mathbf{K}^e \cdot \mathbf{u}$$

Recall: $\mathbf{K}^e$

$$\mathbf{K}_{el} = \frac{E}{\Delta x} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\Rightarrow \frac{dL}{d\alpha_e} > 0$

Move in Direction Opposite to Gradient $\Rightarrow$ Decrease α

Crane

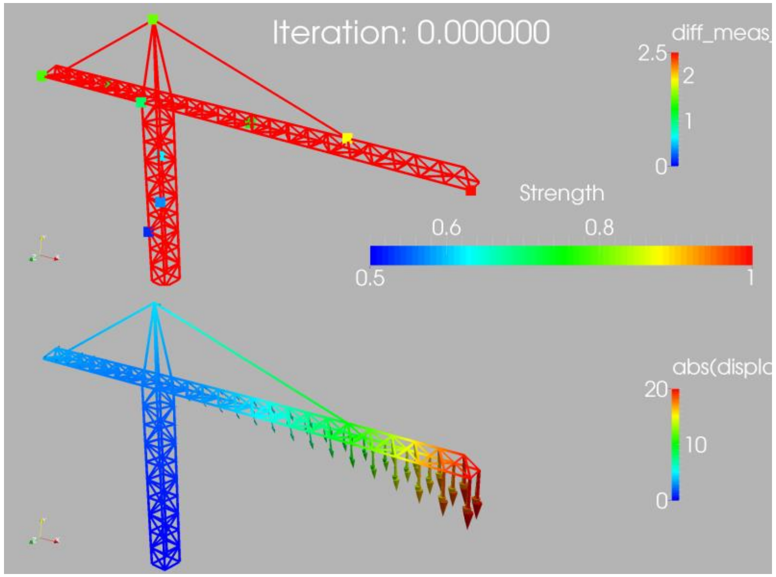
- Truss Elements
- Forces at Extreme Ends of Arm
- 10 Measurement Points
  - Displacements
  - Strains
- Smoothing: Simple Averaging
  - Element → Point → Element [No Volume Considerations]
  - Separate Smoothing of Gradients and Strength Factors

- FEELAST

Crane: 1 Load; <u>Displacements</u>

Weakened Base

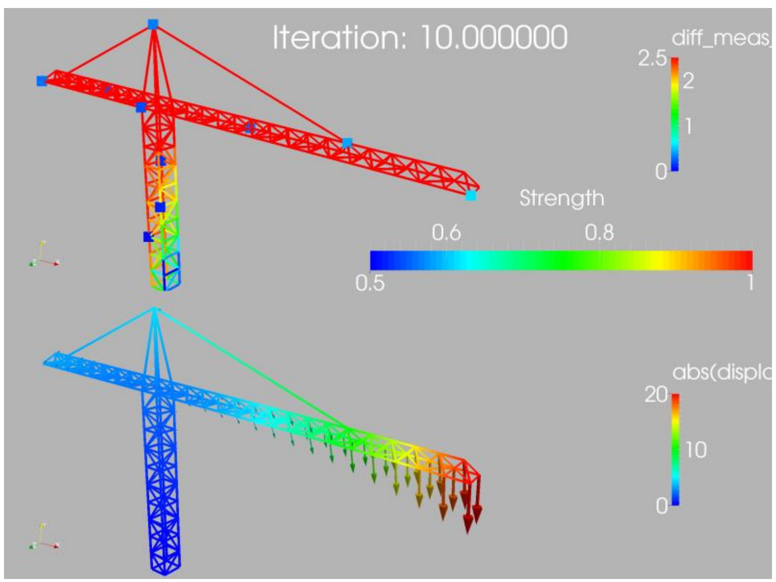
Grad Smoothing, DOF: Base

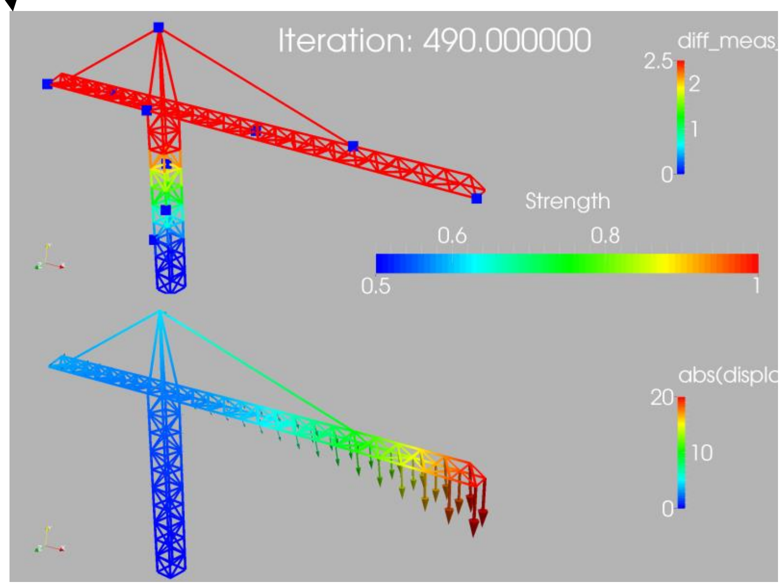


Iteration 0

Iteration 10

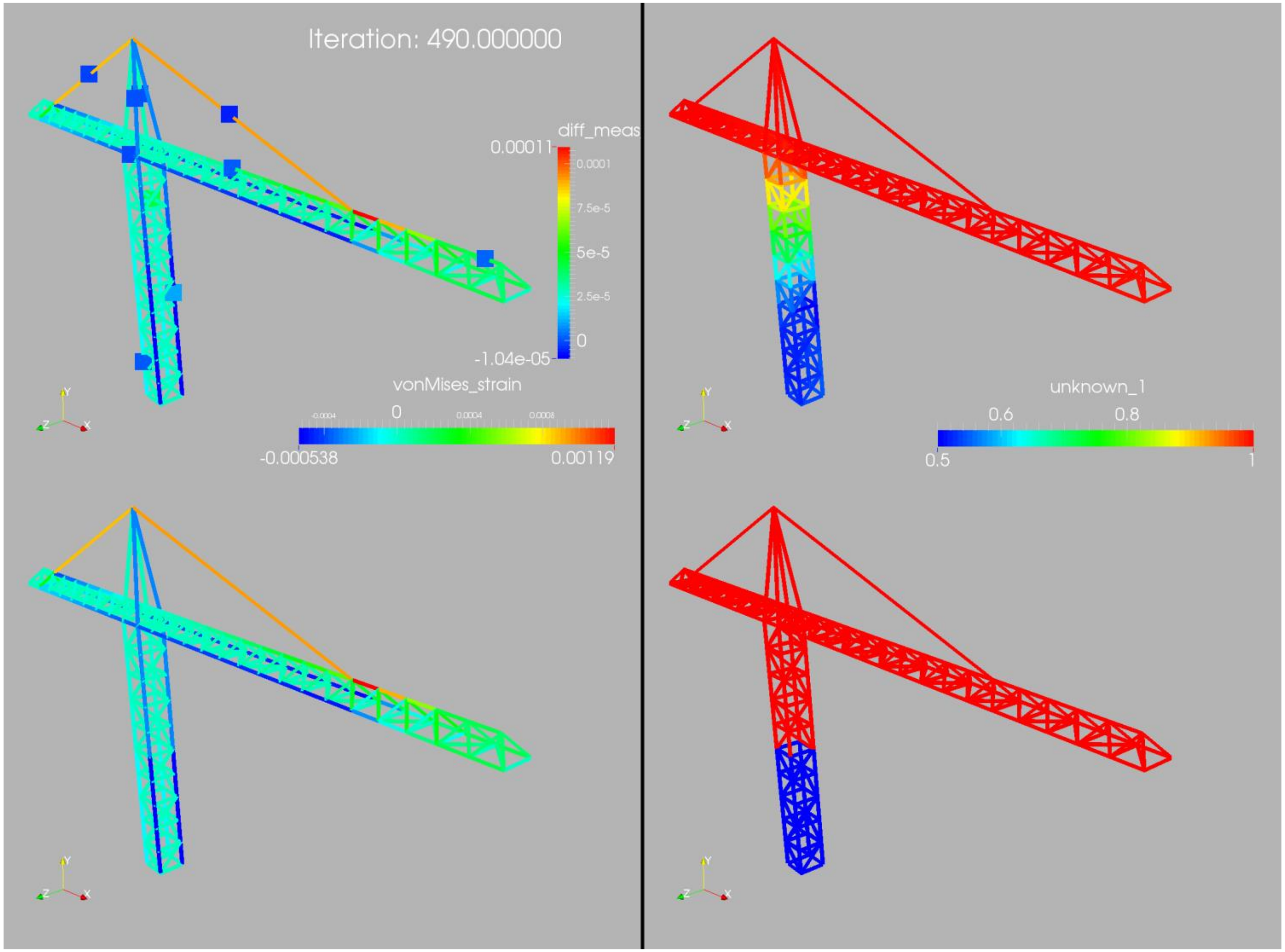
Final





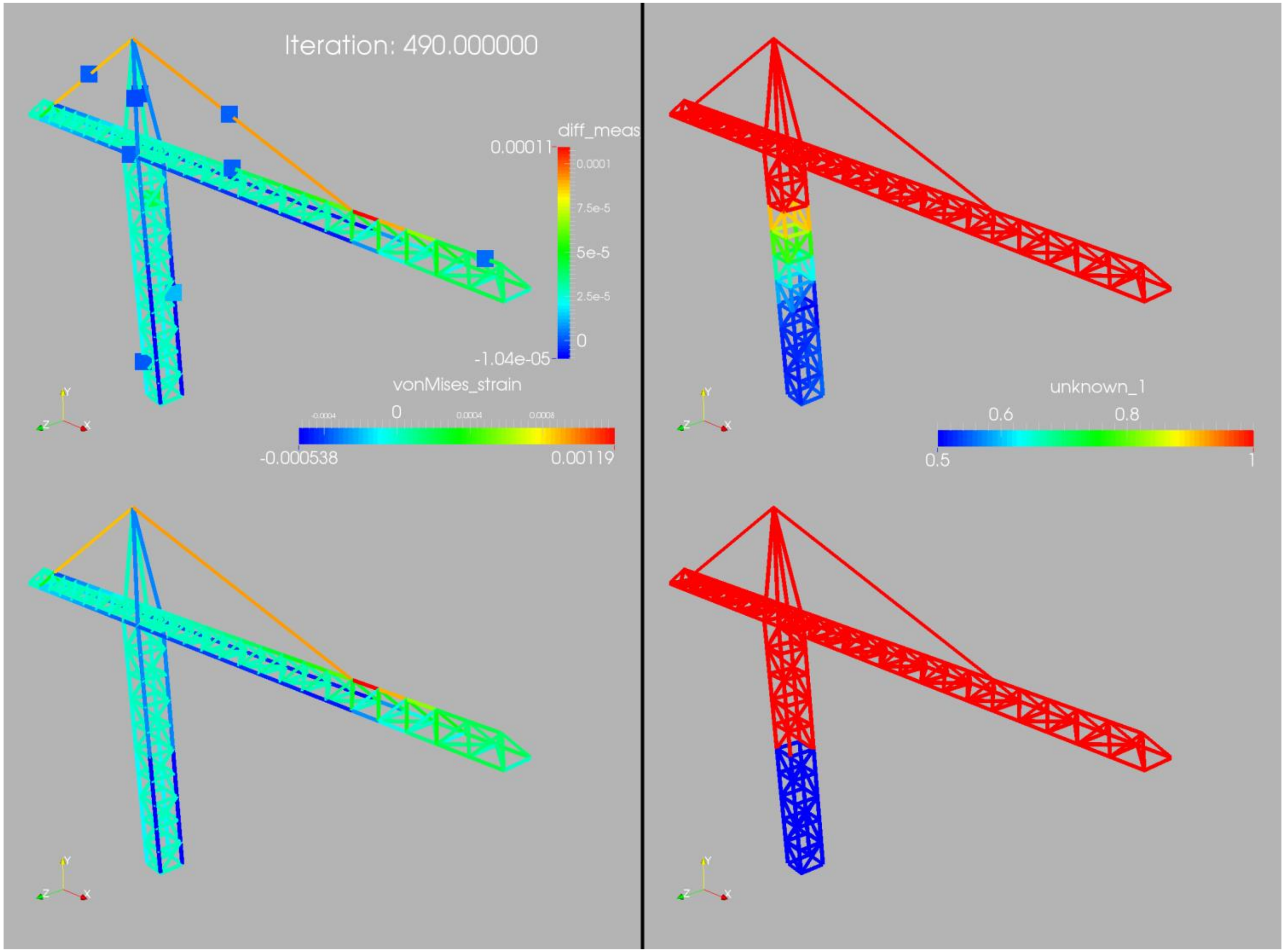
Crane: 1 Load, <u>Strains</u>

Grad Smoothing



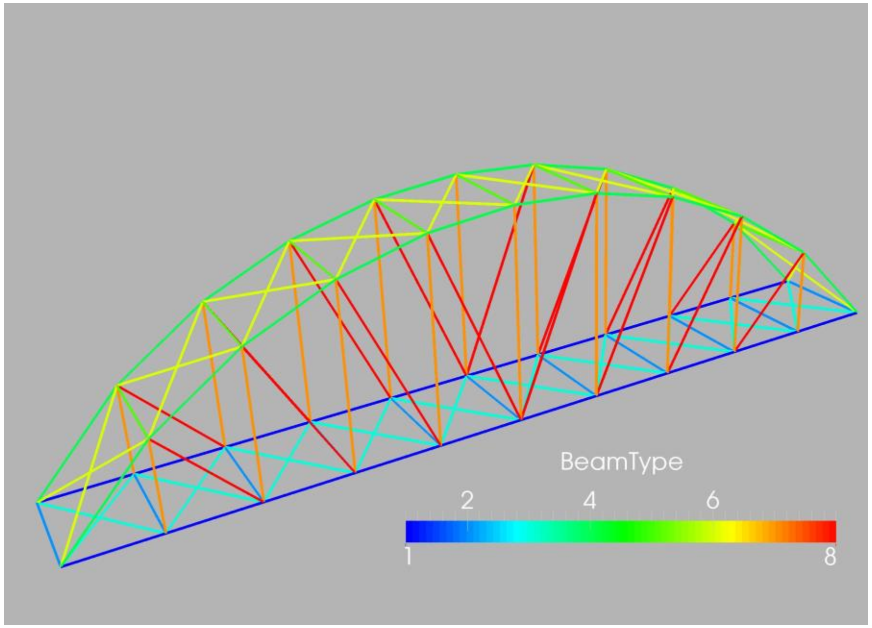
Crane: 1 Load, Strains

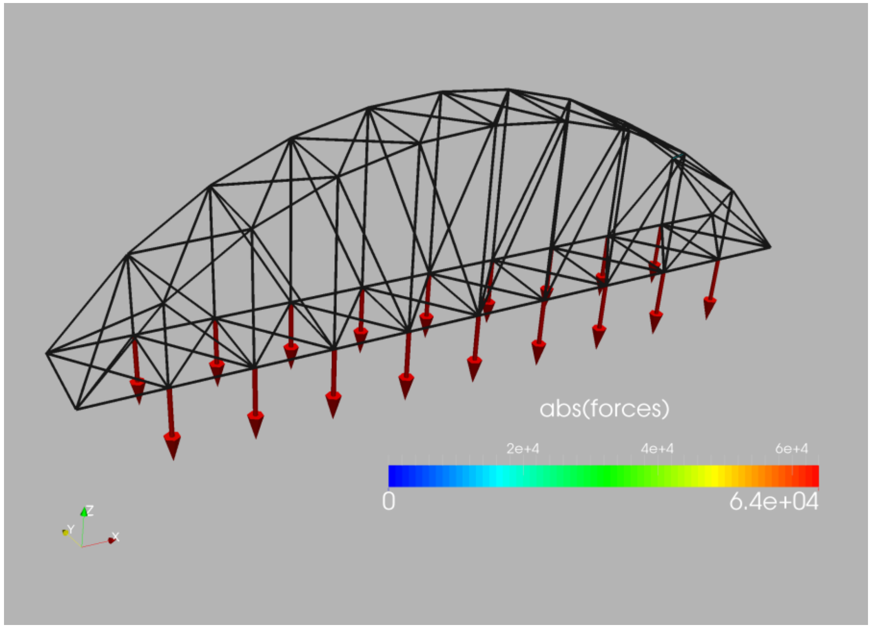
Grad Smoothing, DOF: Tower Only



Bridge

Dimensions: 40x5x10m

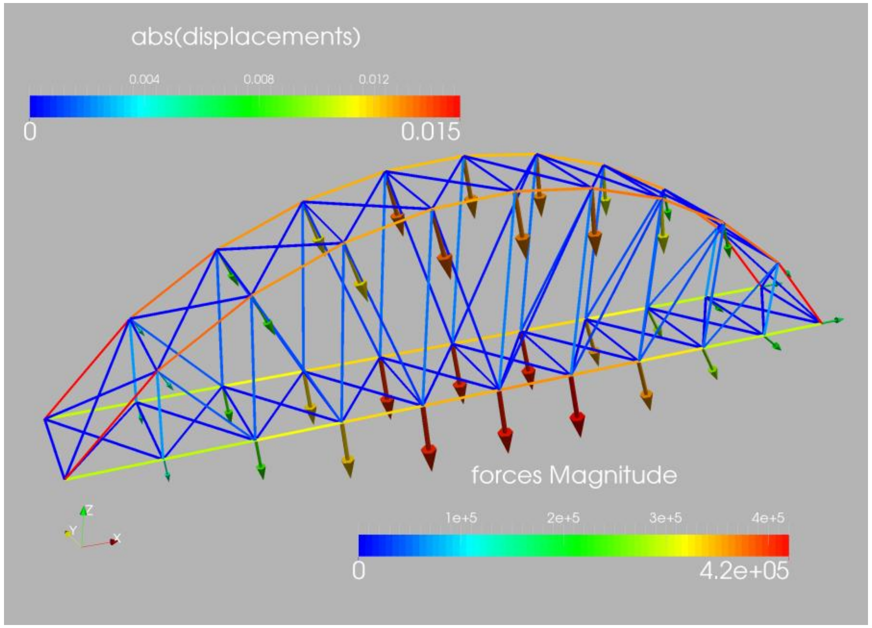


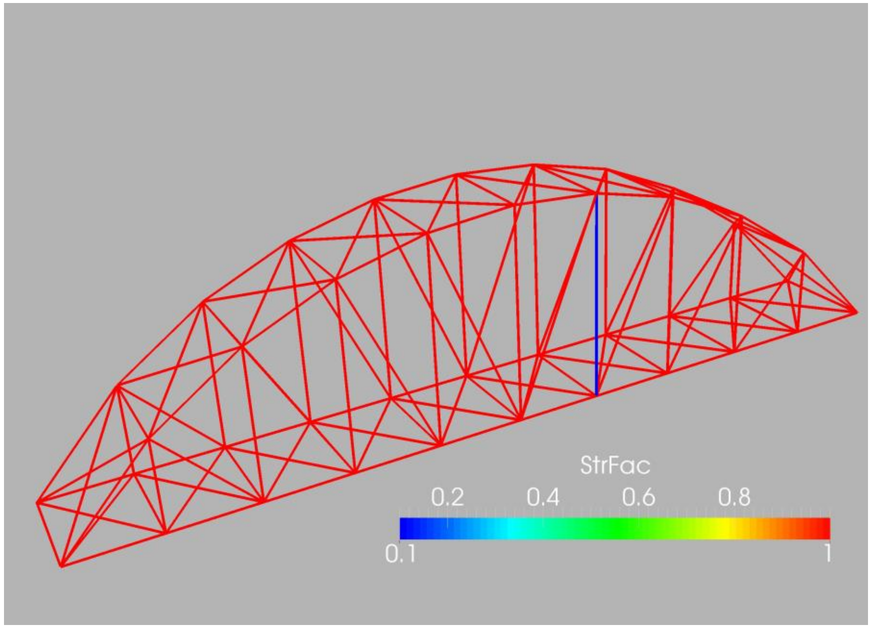


Material: Steel
Trusses: A=1-100cm^2
FEELAST

Forces [+Gravity]

Bridge

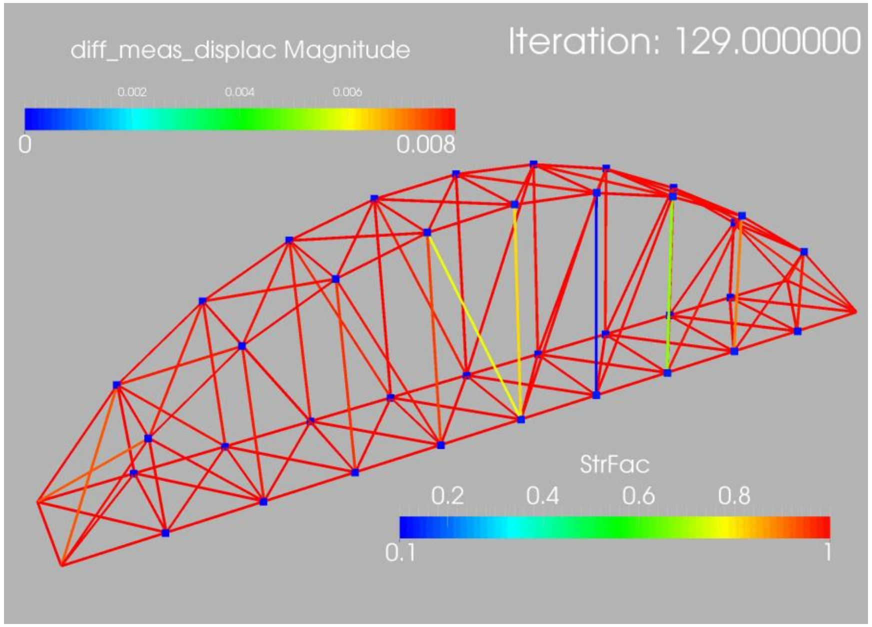


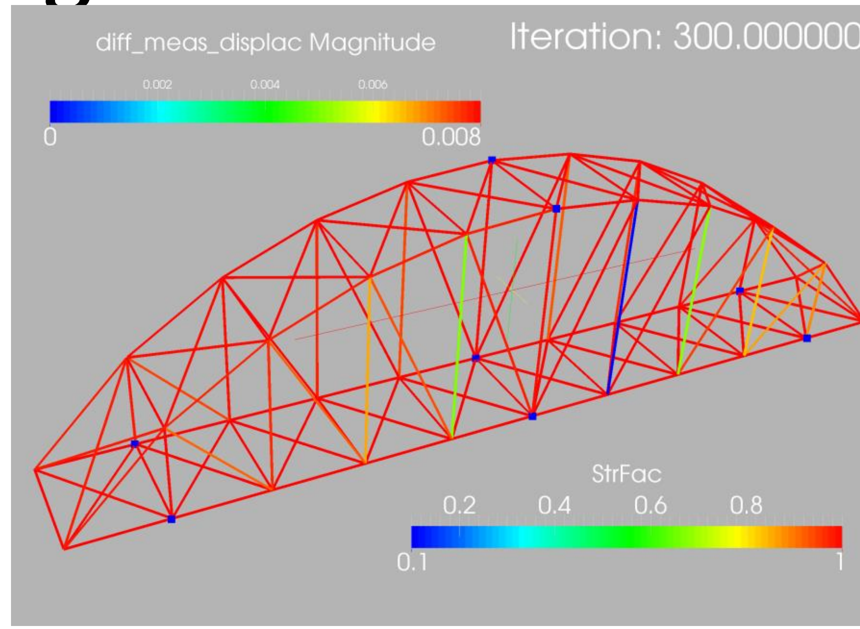


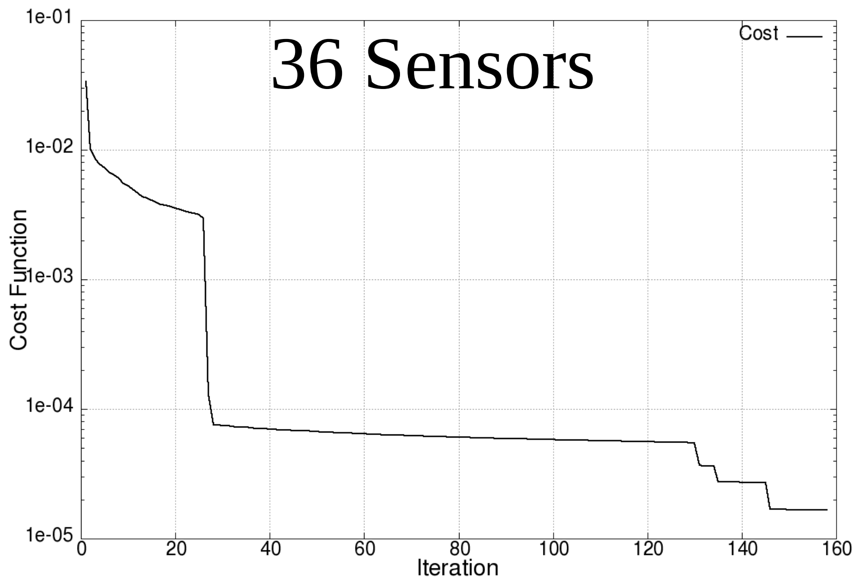
Displacements + Int Forces

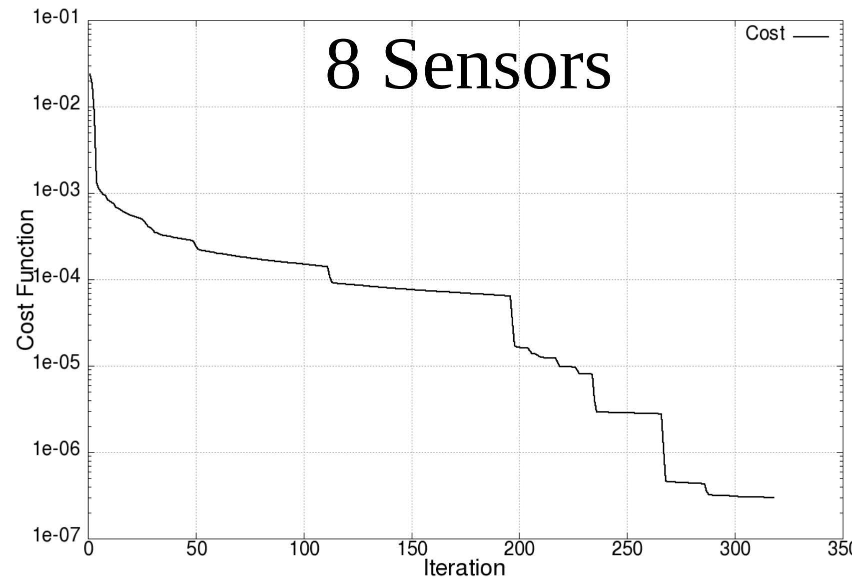
Target StrFac

Bridge

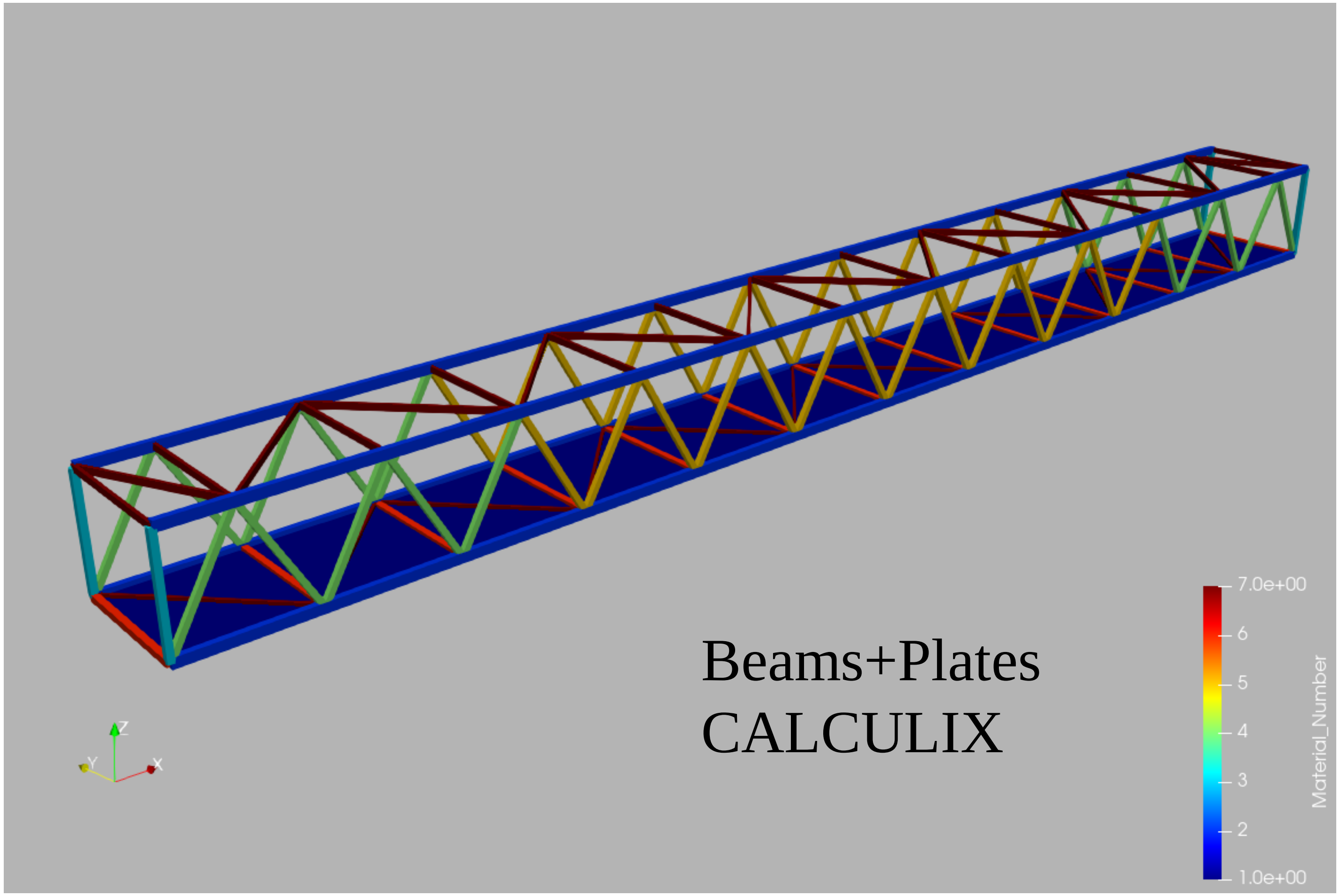

diff_meas_displac Magnitude
Iteration: 129.000000
0.002
0.004
0.006
0
0.008
StrFac
0.2
0.4
0.6
0.8
0.1
1


diff_meas_displac Magnitude
Iteration: 300.000000
0.002
0.004
0.006
0
0.008
StrFac
0.2
0.4
0.6
0.8
0.1
1


36 Sensors
Cost
Cost Function
1e-01
1e-02
1e-03
1e-04
1e-05
0
20
40
60
80
100
120
140
160
Iteration

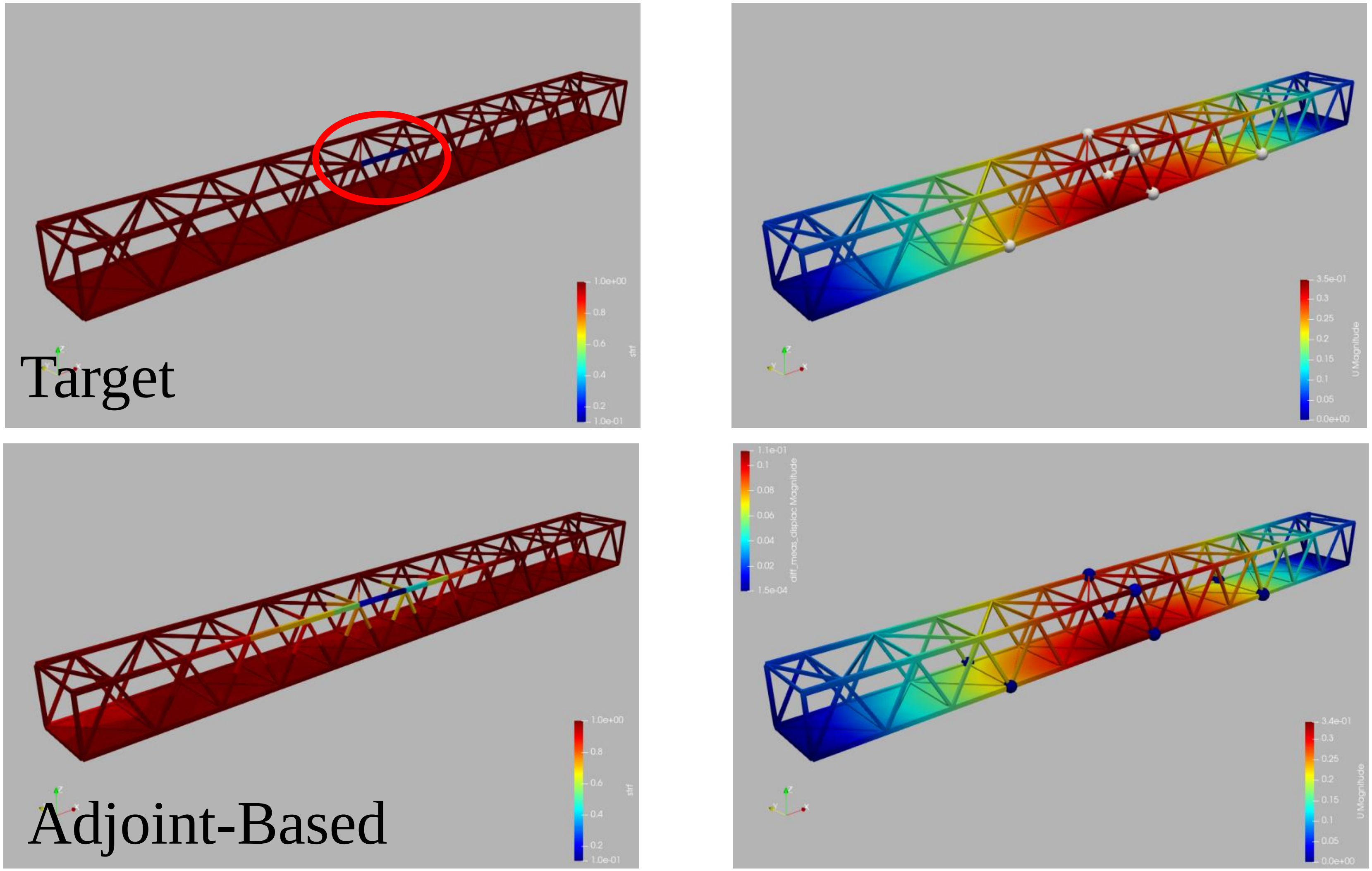

8 Sensors
Cost
Cost Function
1e-01
1e-02
1e-03
1e-04
1e-05
1e-06
1e-07
0
50
100
150
200
250
300
350
Iteration

Footbridge

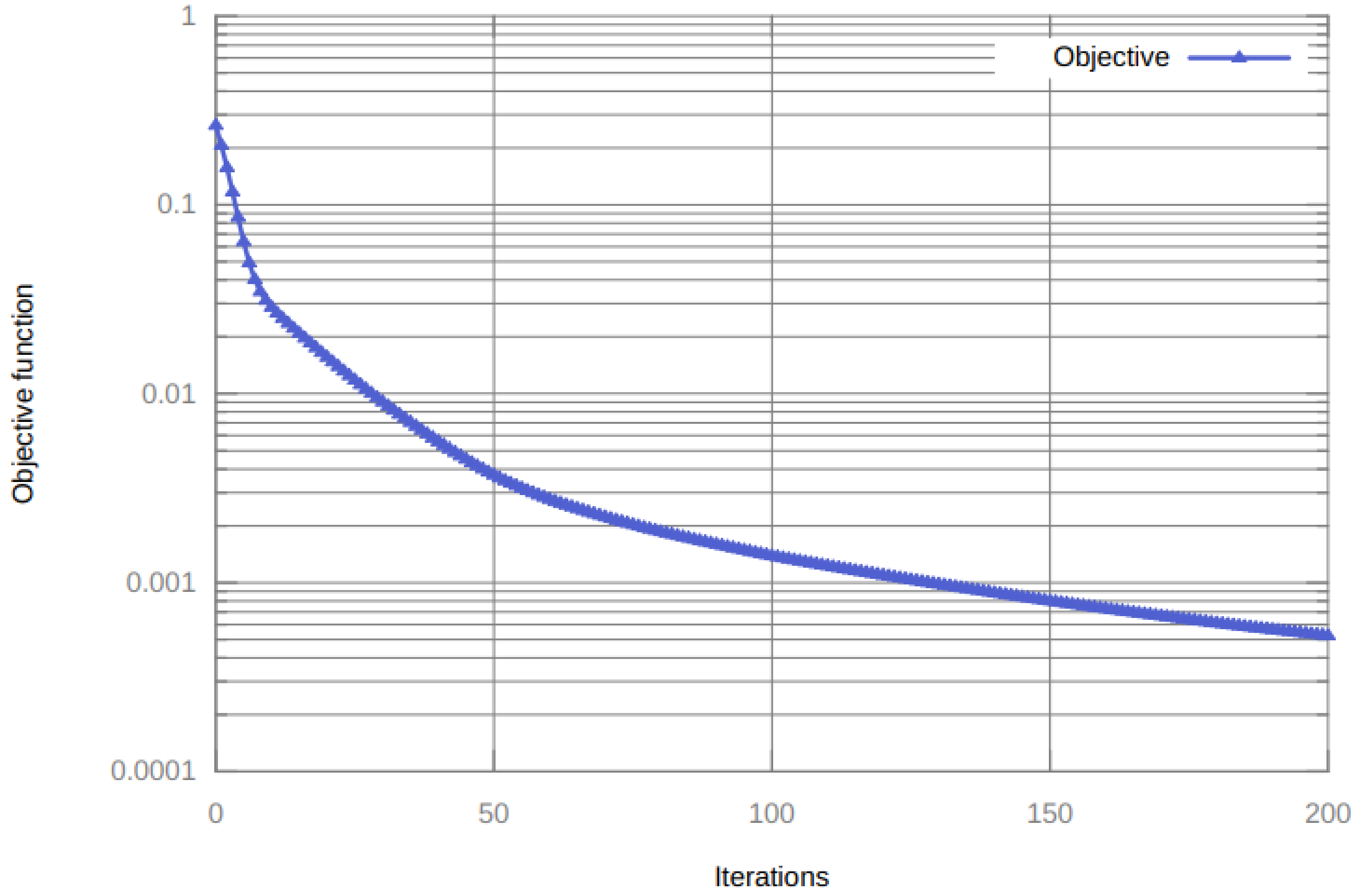


Beams+Plates
CALCULIX
7.0e+00
6
5
4
3
2
1.0e+00
Material_Number
Z
Y
X

Footbridge



Footbridge



Objective
Objective function
1
0.1
0.01
0.001
0.0001
0
50
100
150
200
Iterations

Plate With Hole

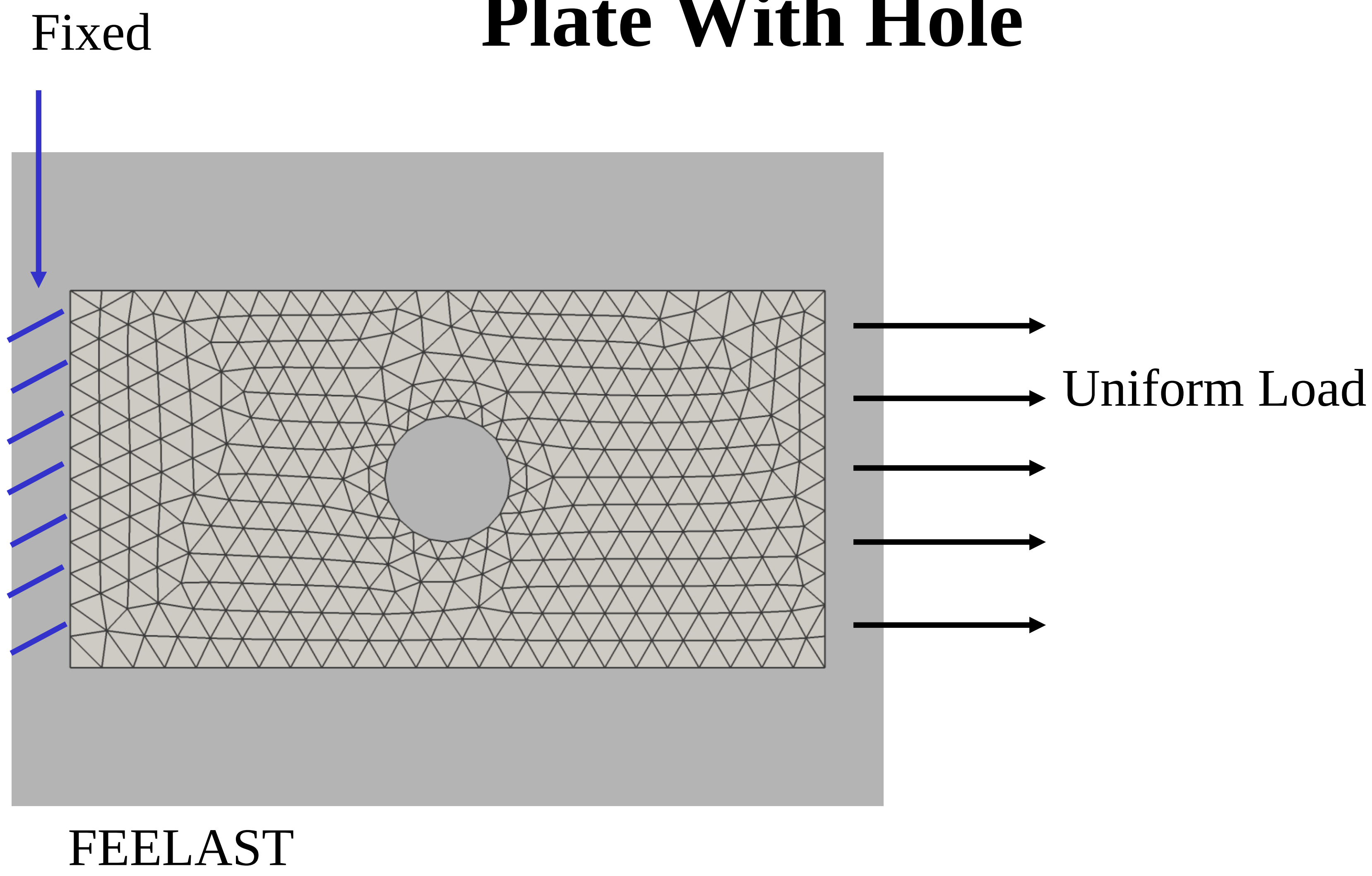


Plate With Hole:
Small Damaged Region

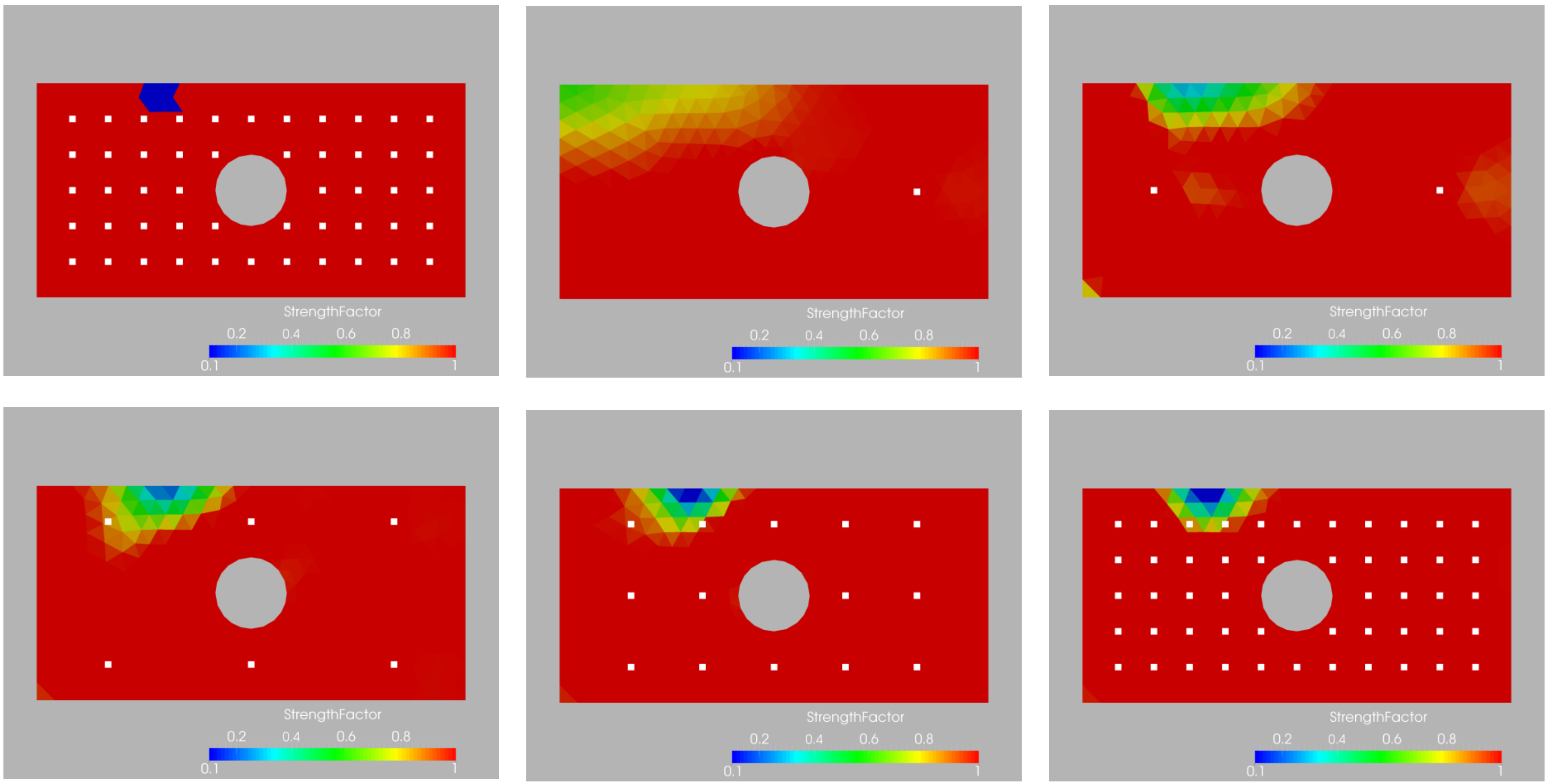
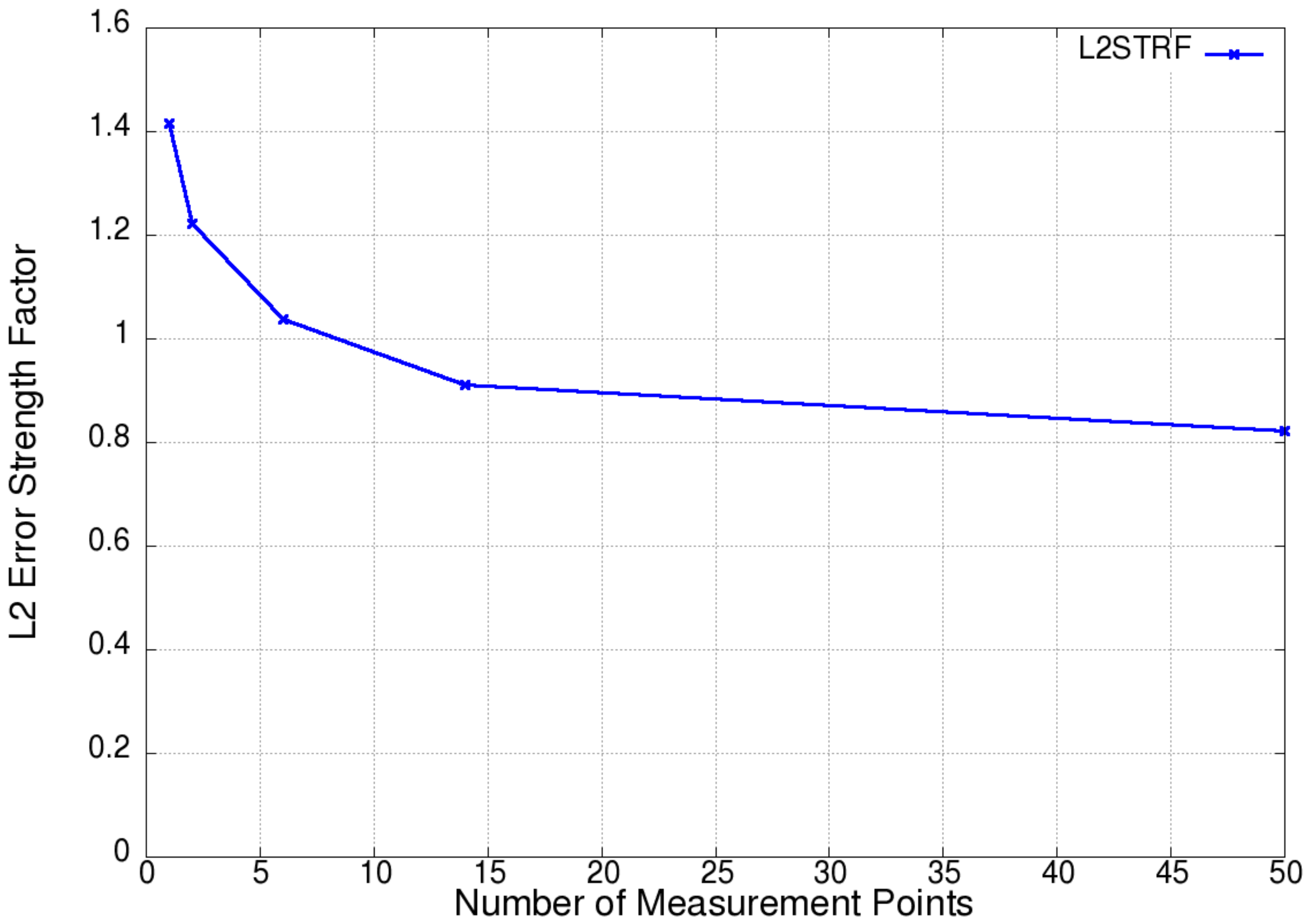
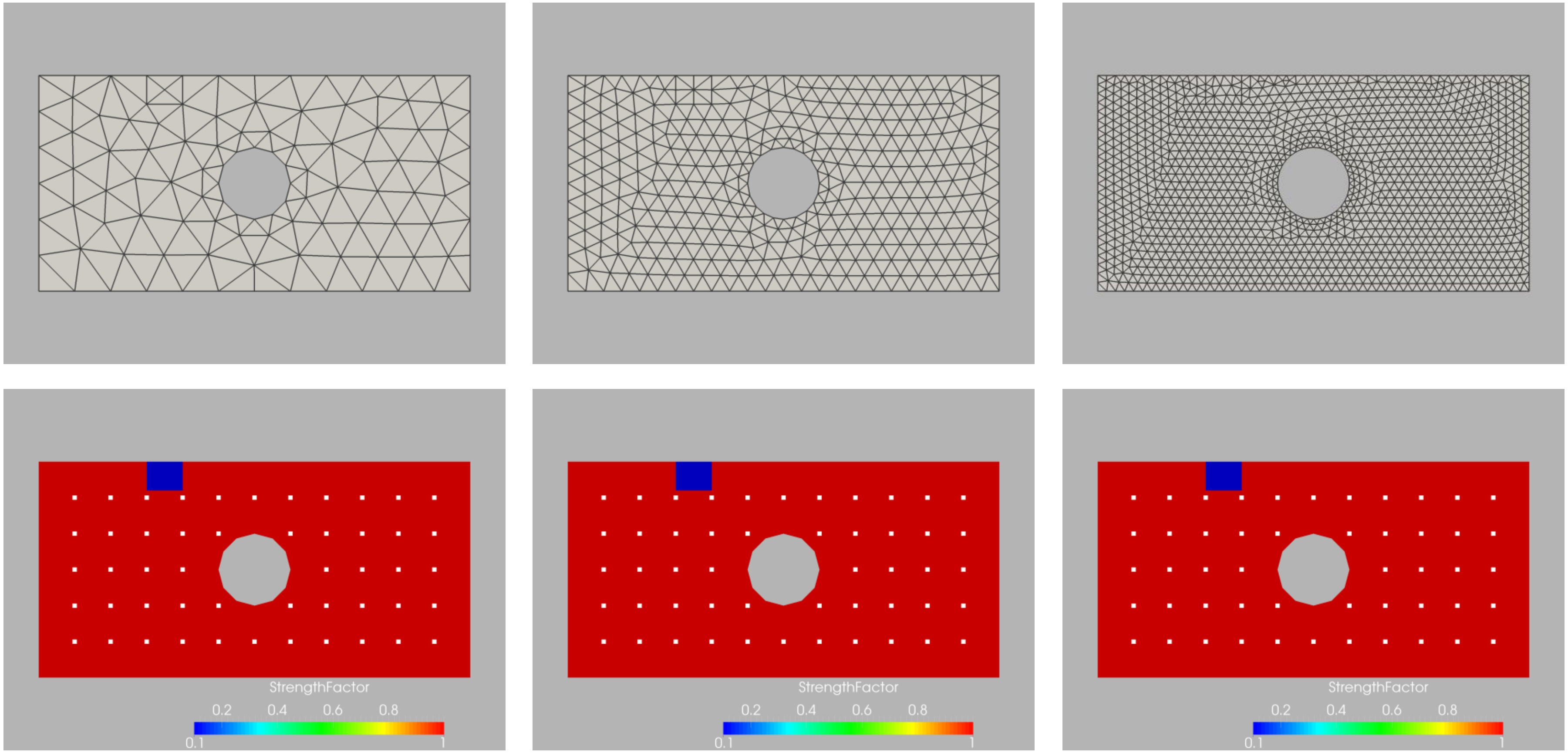
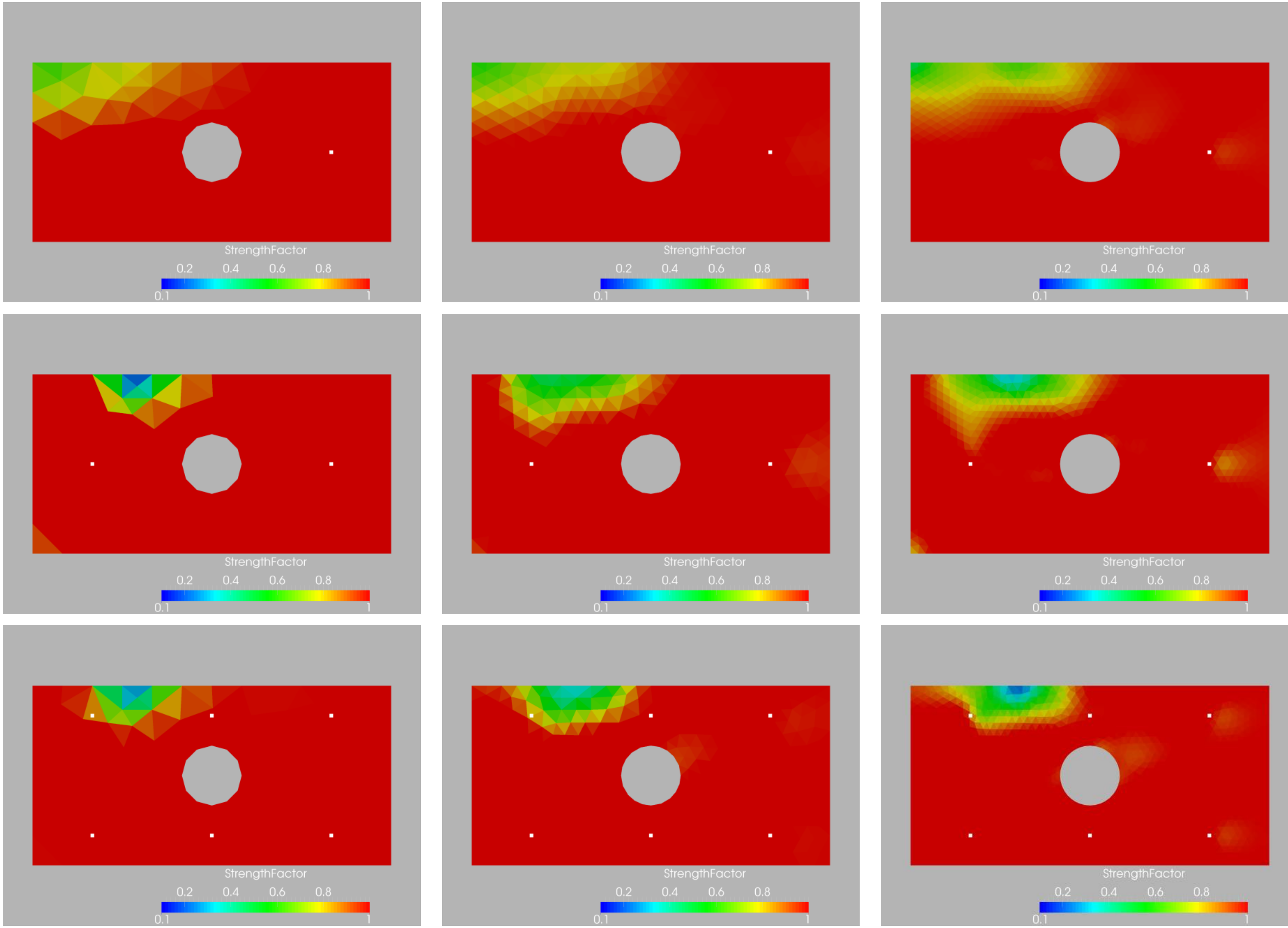


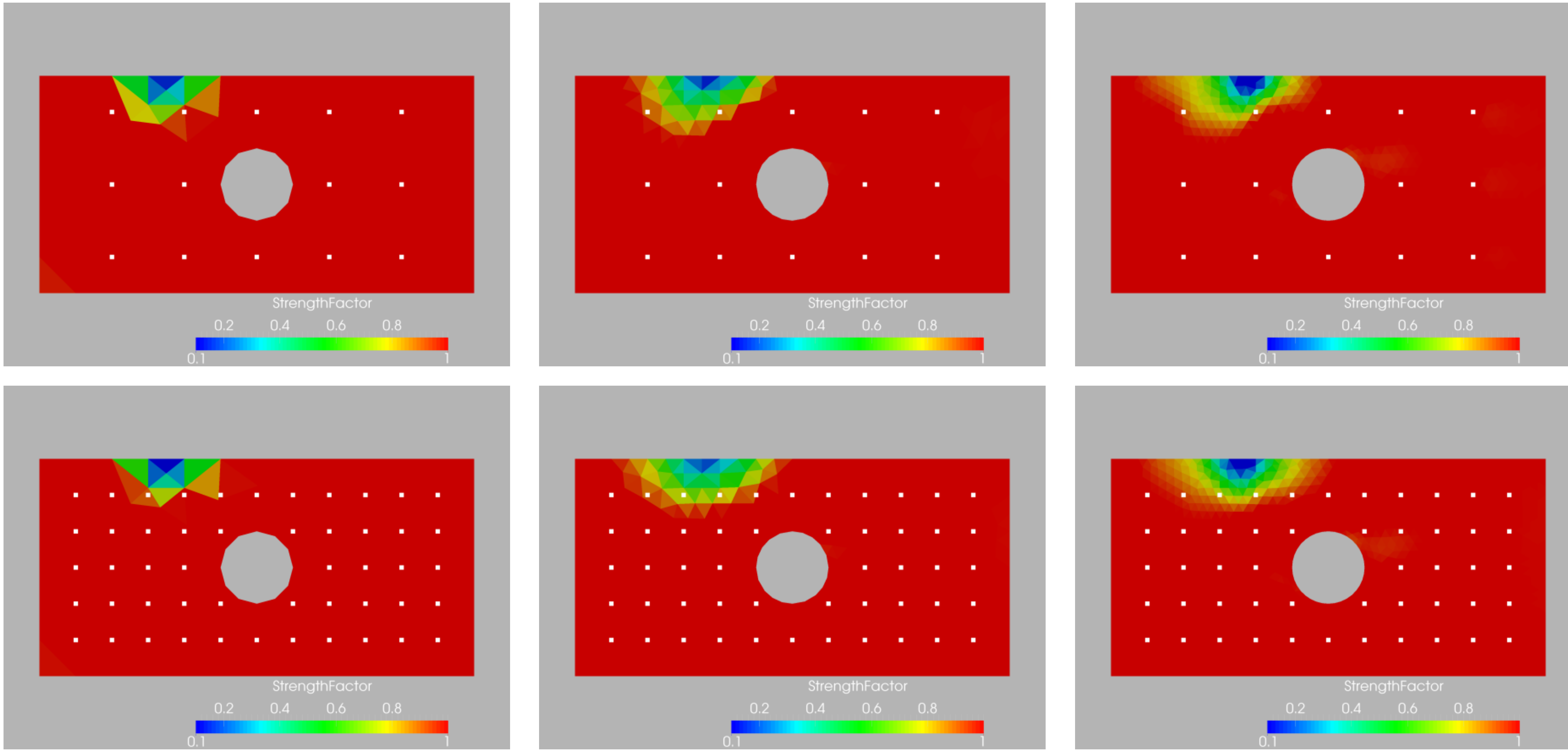
Plate With Hole

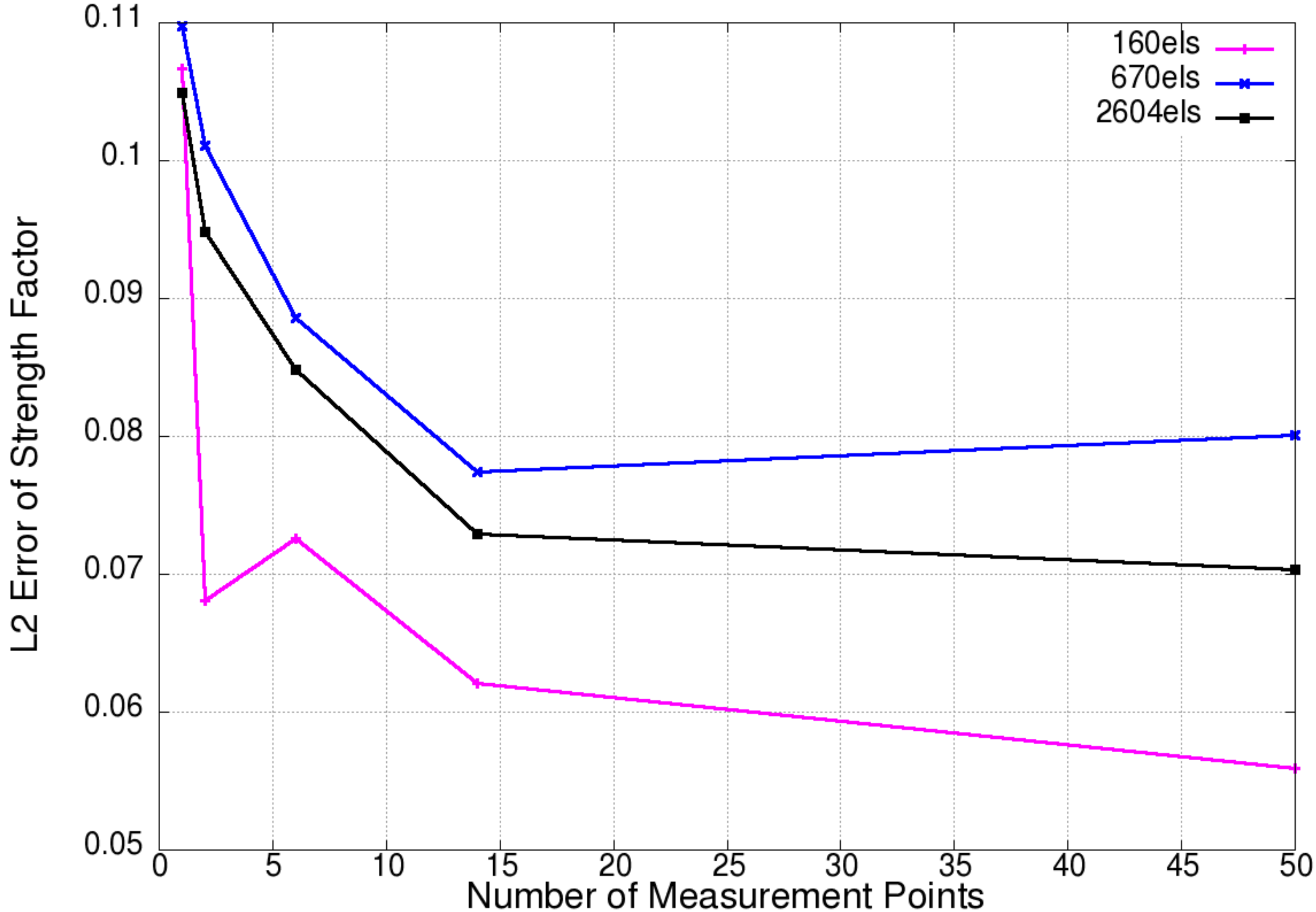


Convergence Study




StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1
StrengthFactor
0.2 0.4 0.6 0.8
0.1
1


StrengthFactor
0.2 0.4 0.6 0.8
0.1
1



160els
670els
2604els
L2 Error of Strength Factor
Number of Measurement Points
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0
5
10
15
20
25
30
35
40
45
50

Plate With Hole 3D

- 16Kels
- 120Kels
- X=0: Clamped
- X=Xmax: Fx
- FEELAST

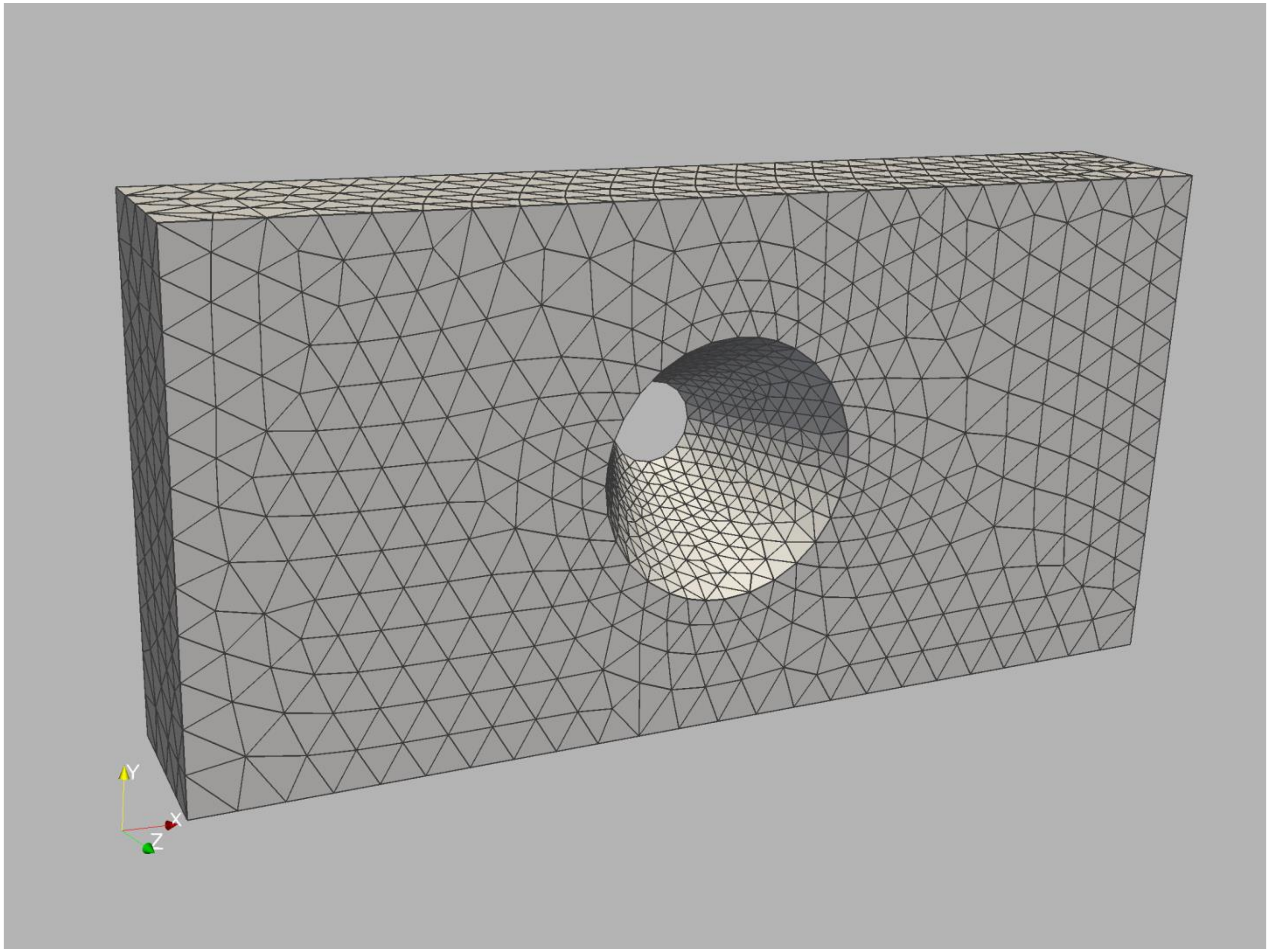


Plate With Hole 3D

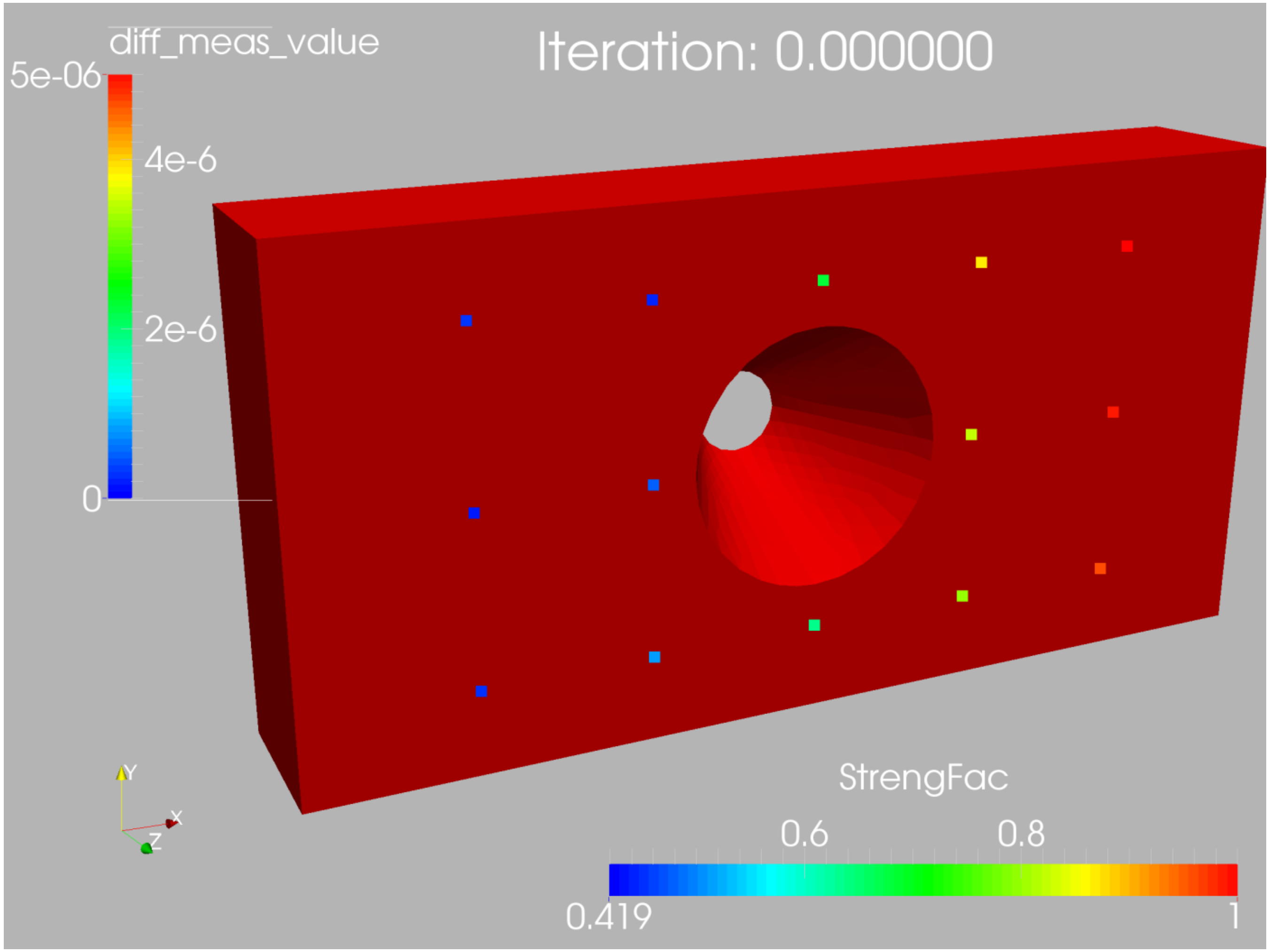
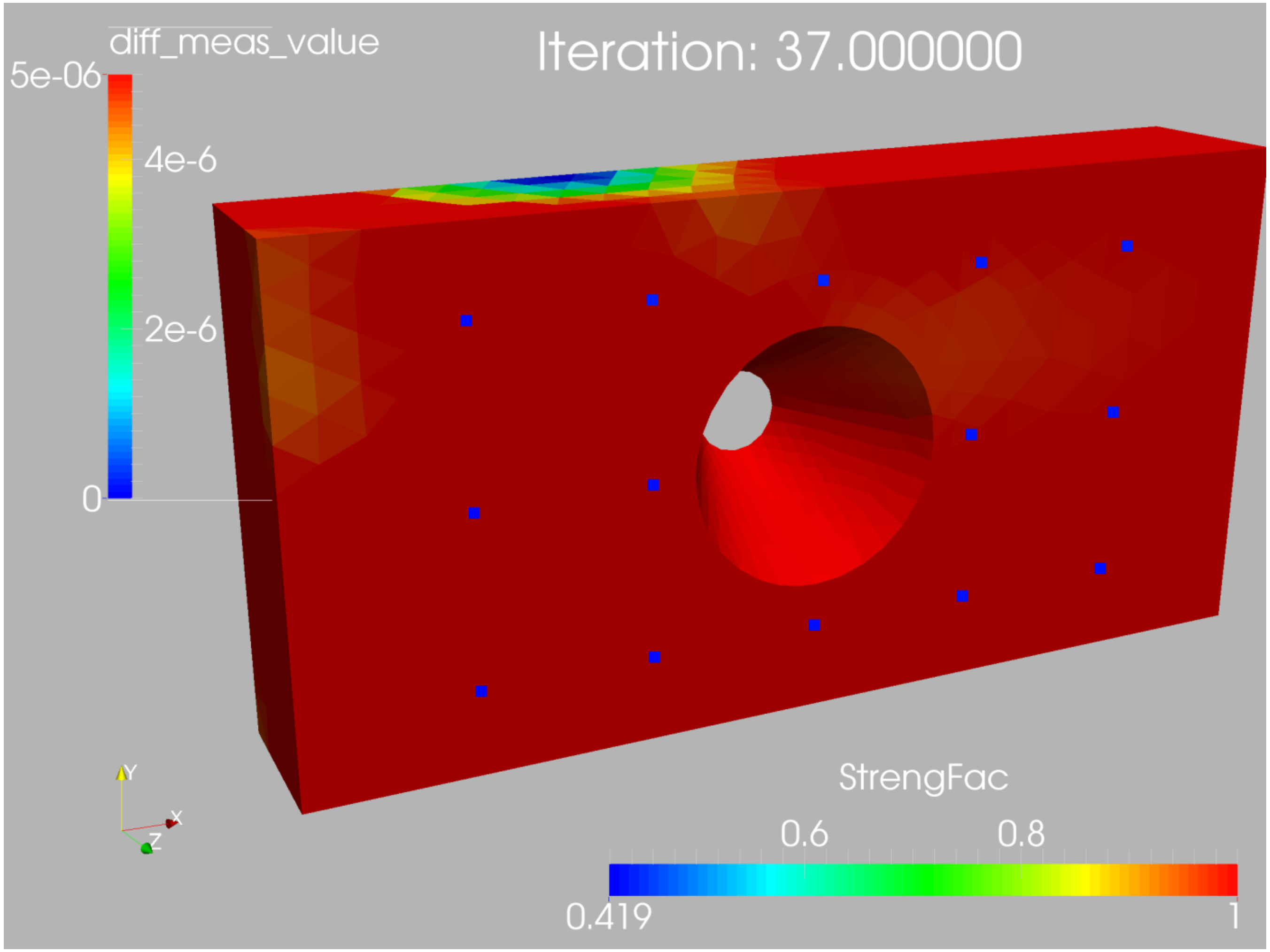
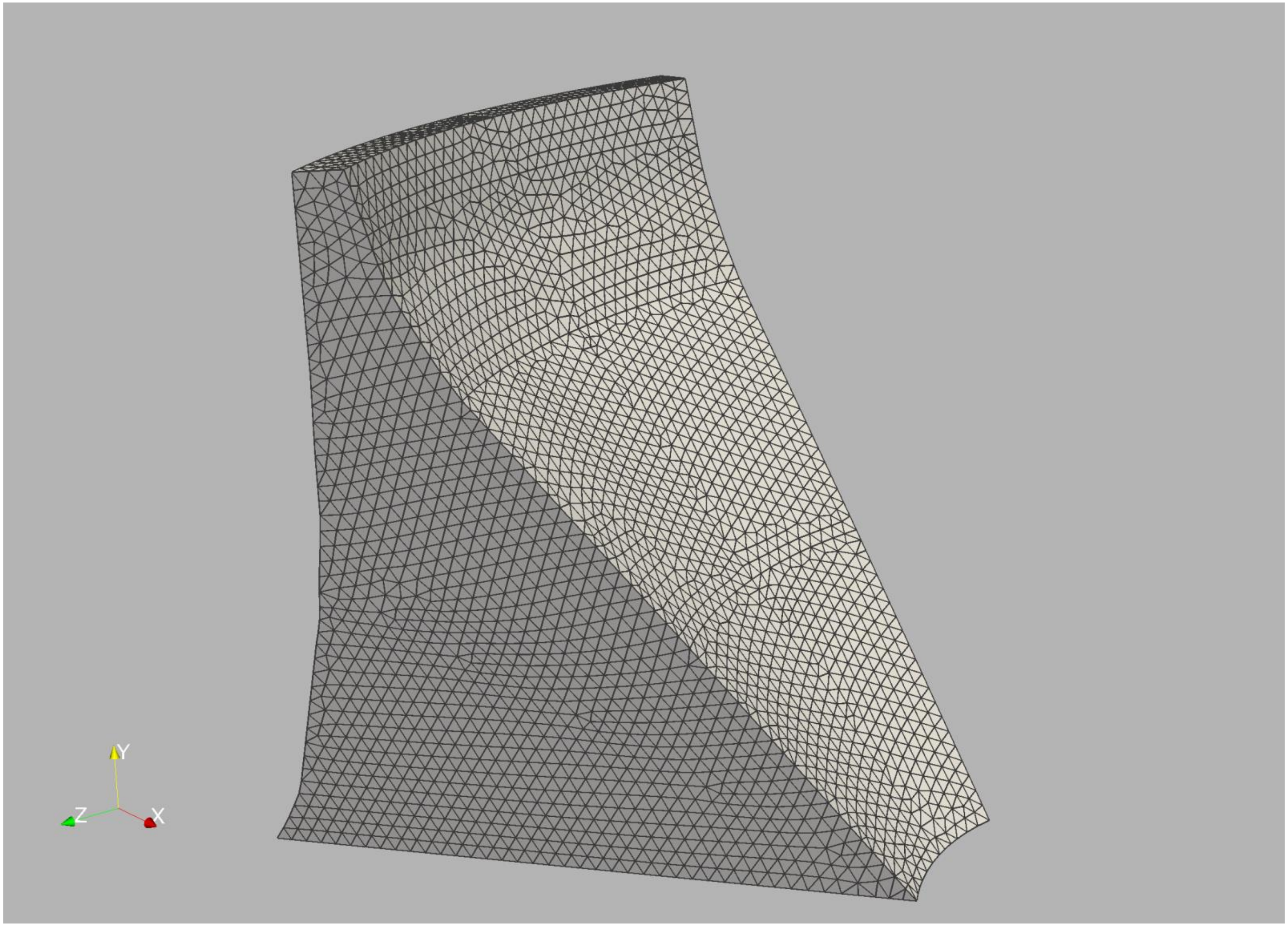


Plate With Hole 3D

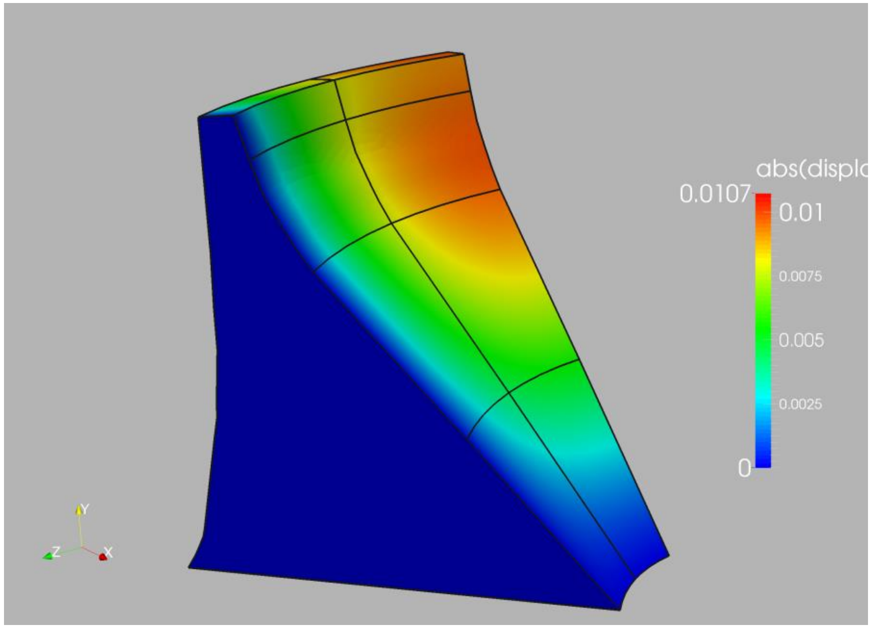


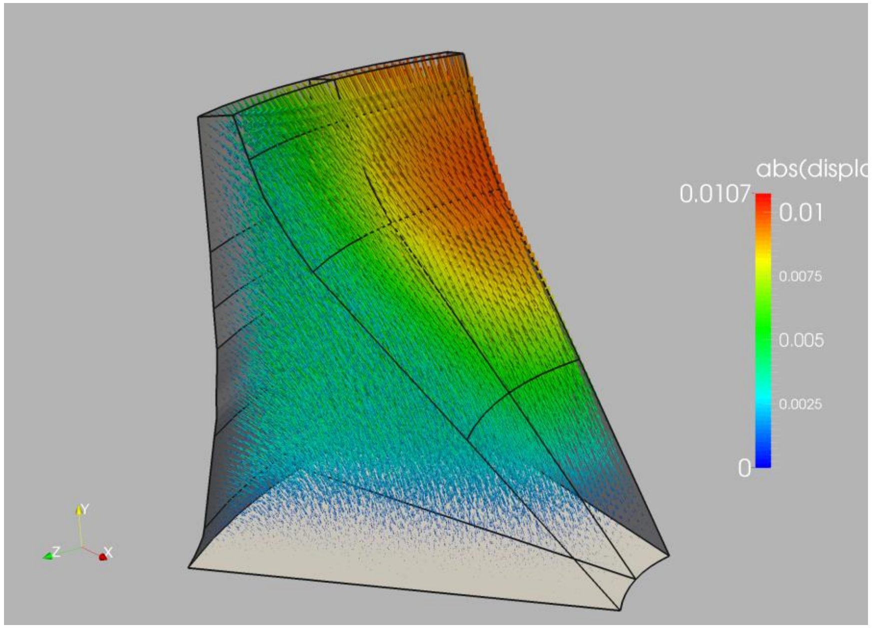
Hoover Dam

Hoover Dam



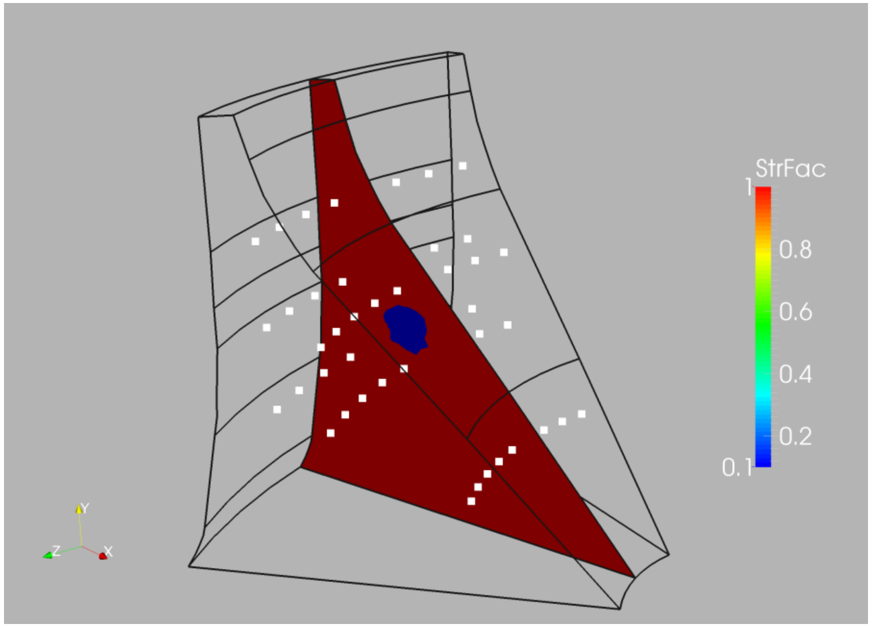
Hoover Dam: 51 Sensors



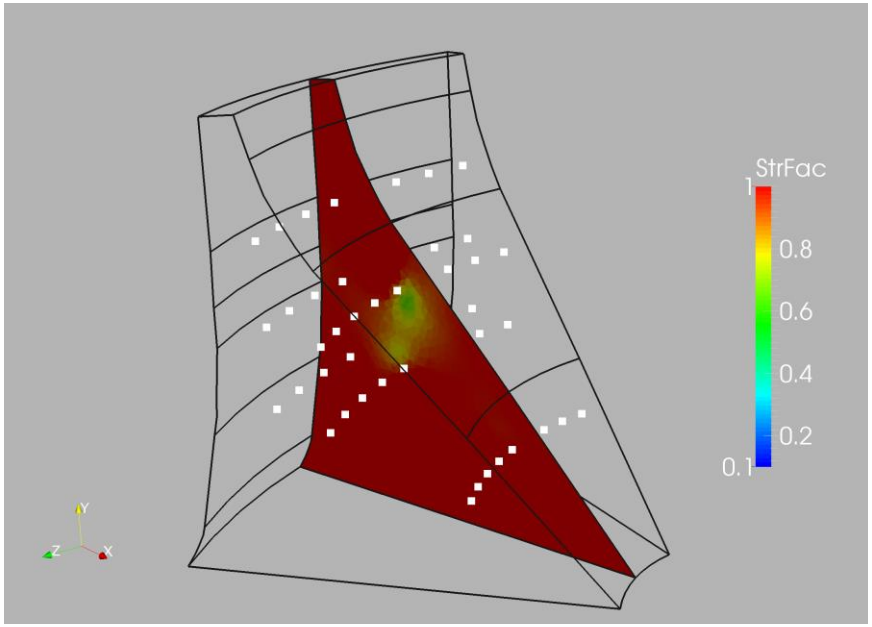


FEELAST

Hoover Dam: 51 Sensors

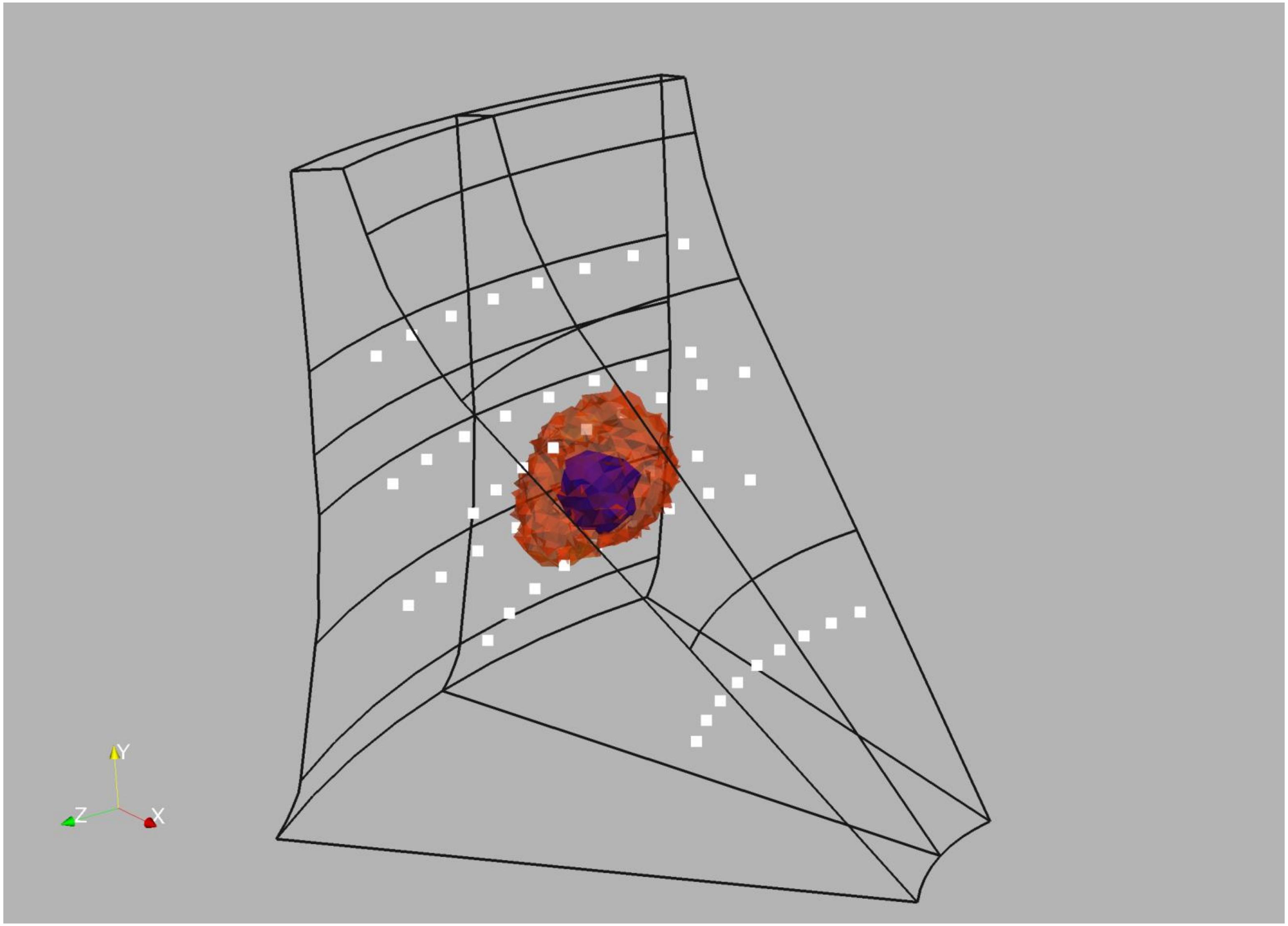


Target



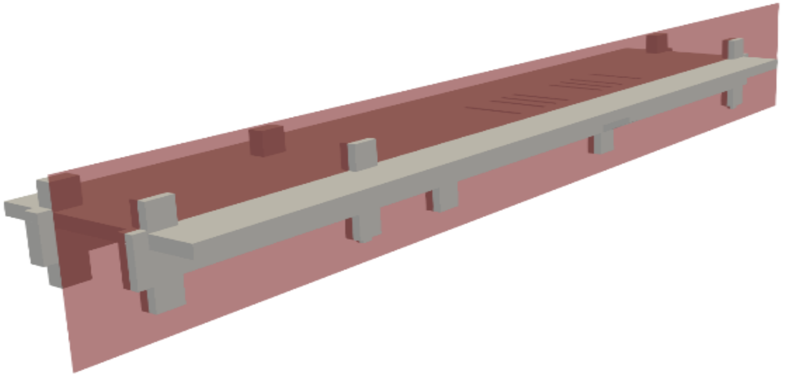
Detected/Recovered

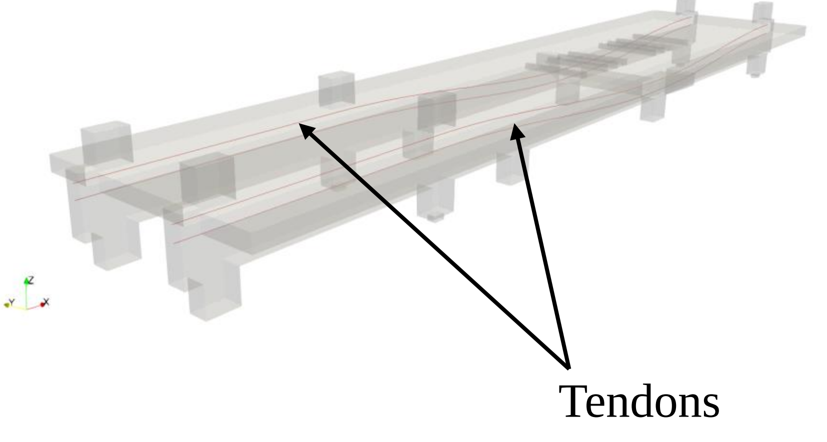
Hoover Dam: 51 Sensors



Concerto Bridge

- Built in 2005 by iBMB – TU Braunschweig for Testing Measurement Technologies
- Loads Can be Applied Using Hydraulic Presses and External Tendons
- Constructed with Prestressed, Post-Tensioned Concrete

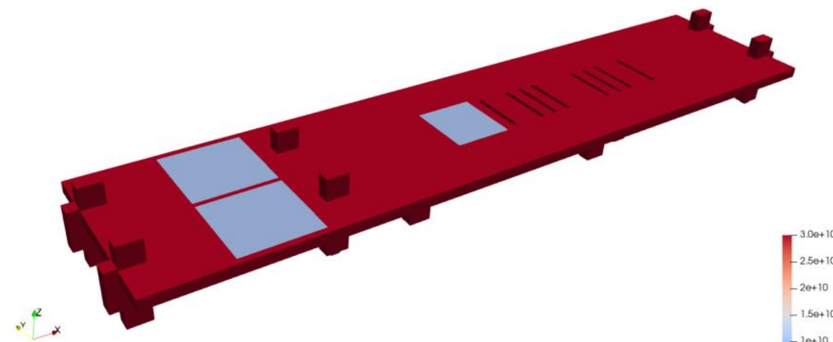




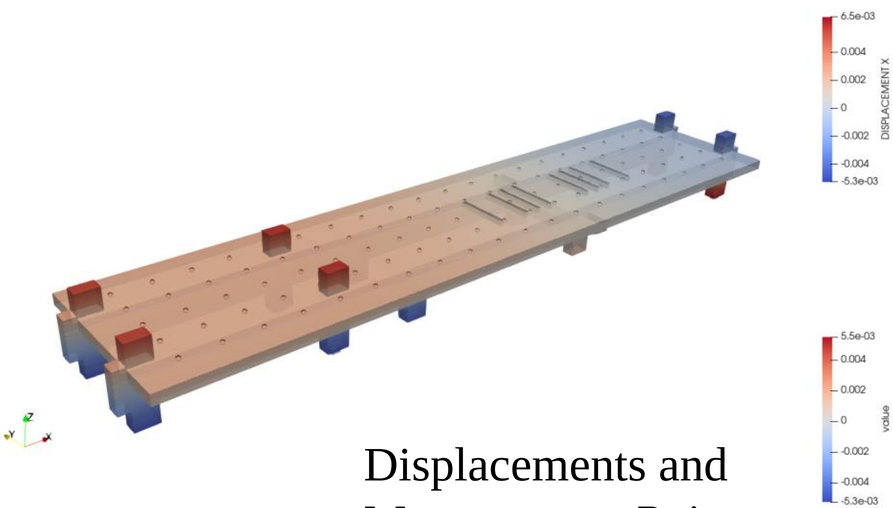
Concerto Bridge

Configuration: KRATOS

- Concrete: 77k Small Displacement Hex Elements
- Tendons: 800 Truss Elements
- 100 Potential Displacement (x) Sensors



Damage Prescribed

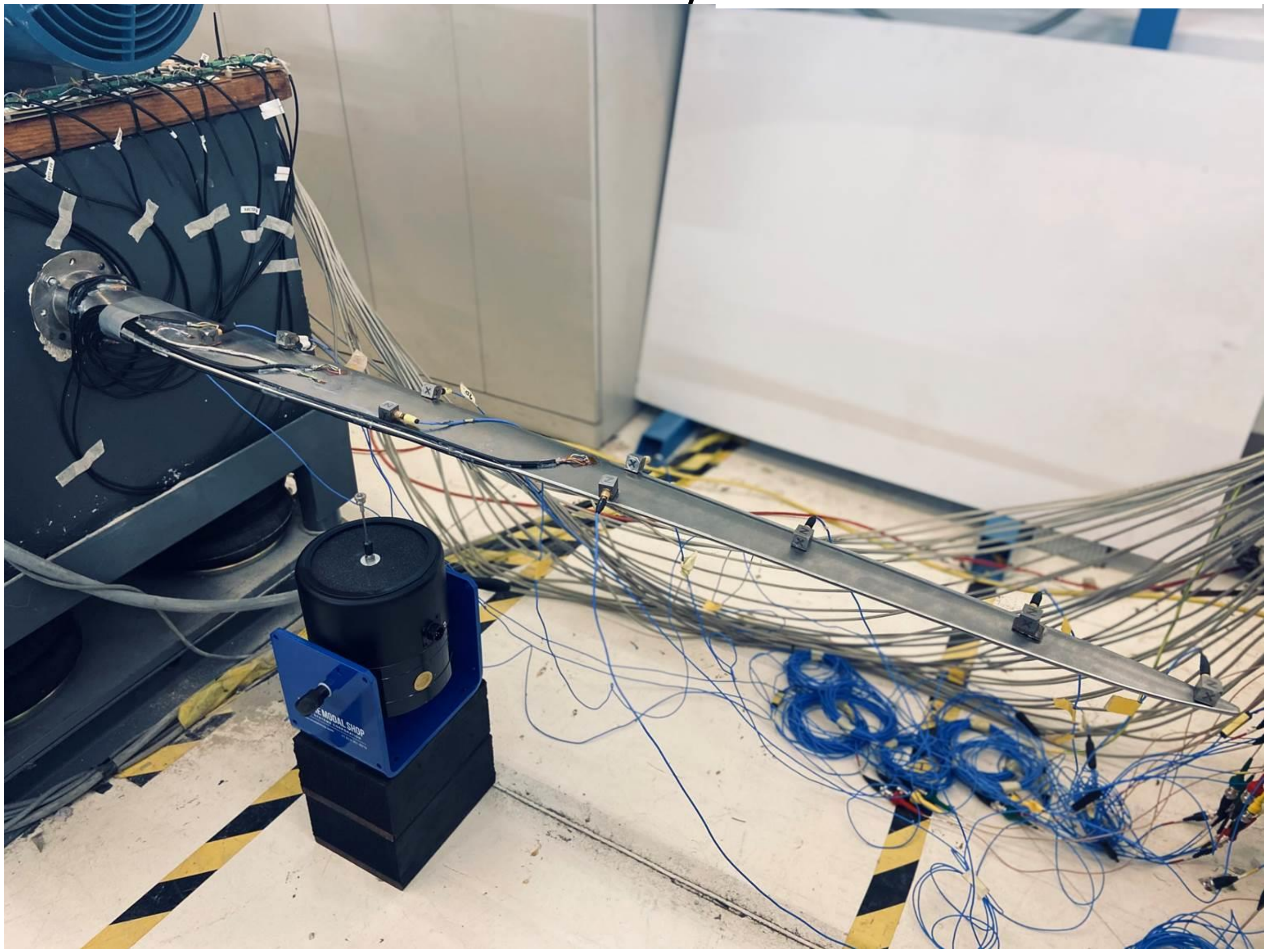


Displacements and Measurement Points

Damage Found

Blade

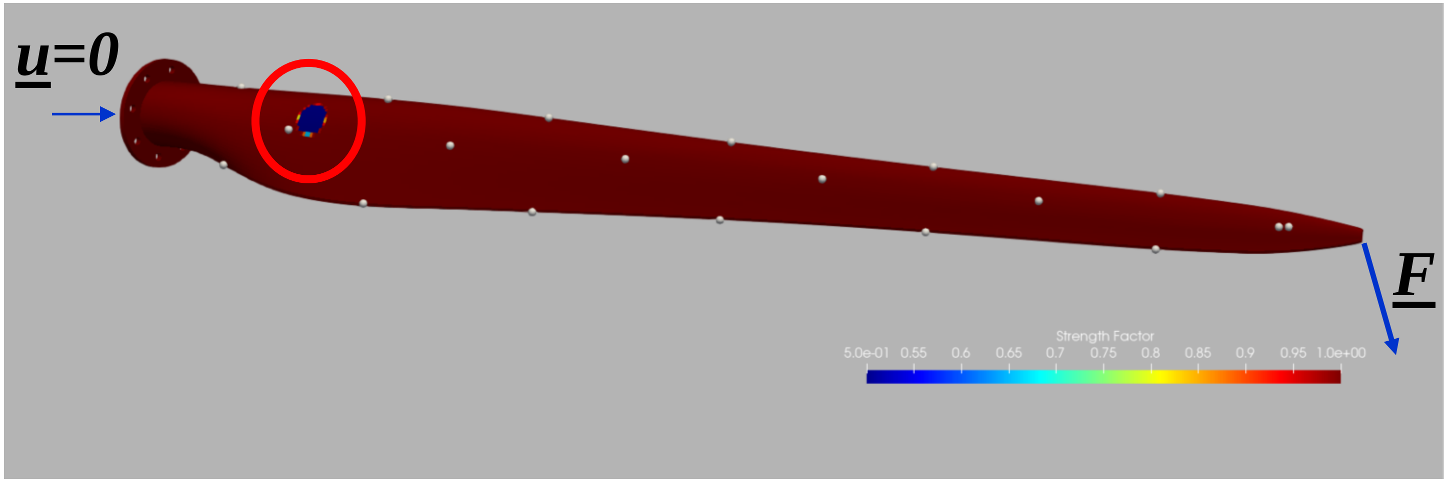
Siemens Project: xDT Titanium Blade
Courtesy S. Vettori and E. Di Lorenzo

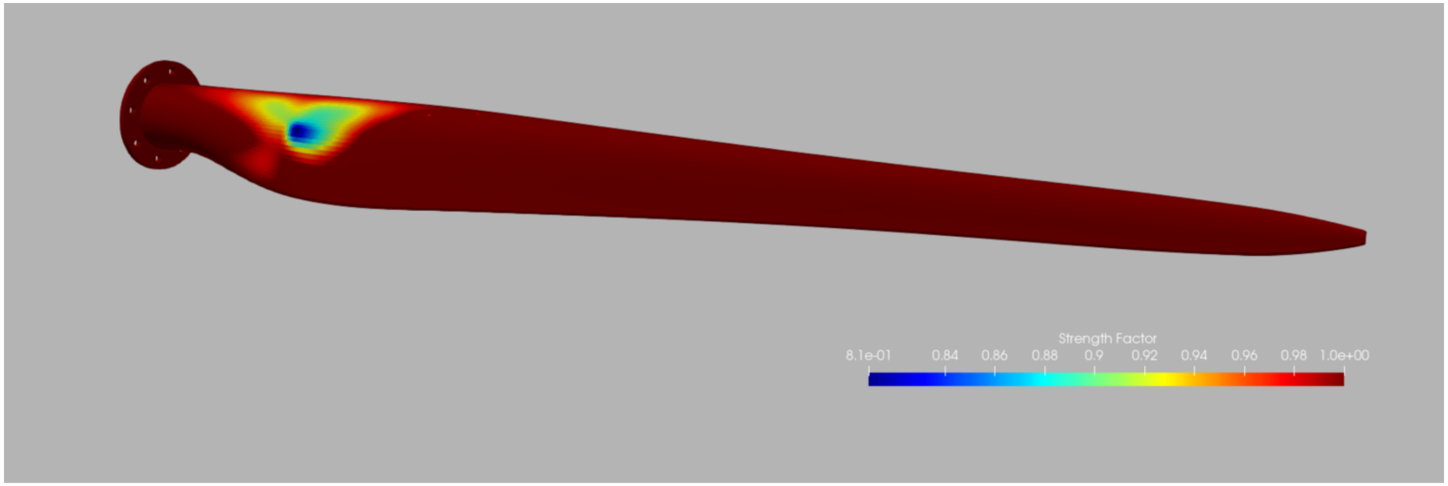


SIEMENS

Blade

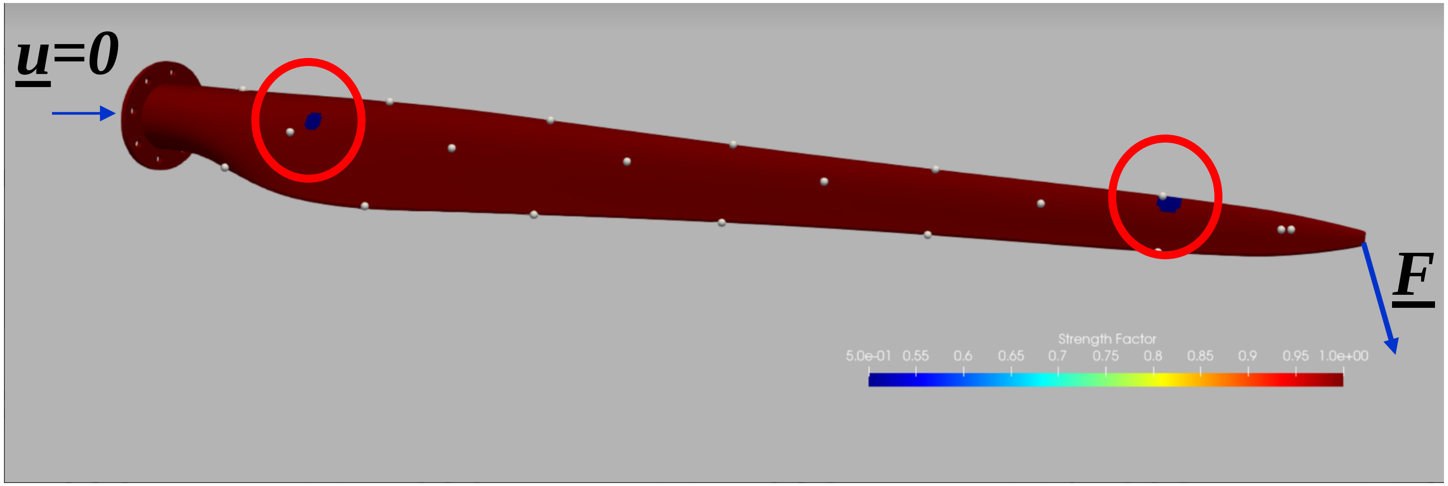
```
nelem=34K (Hex,Pr)
nsens=19
nload= 1
CALCULIX
```

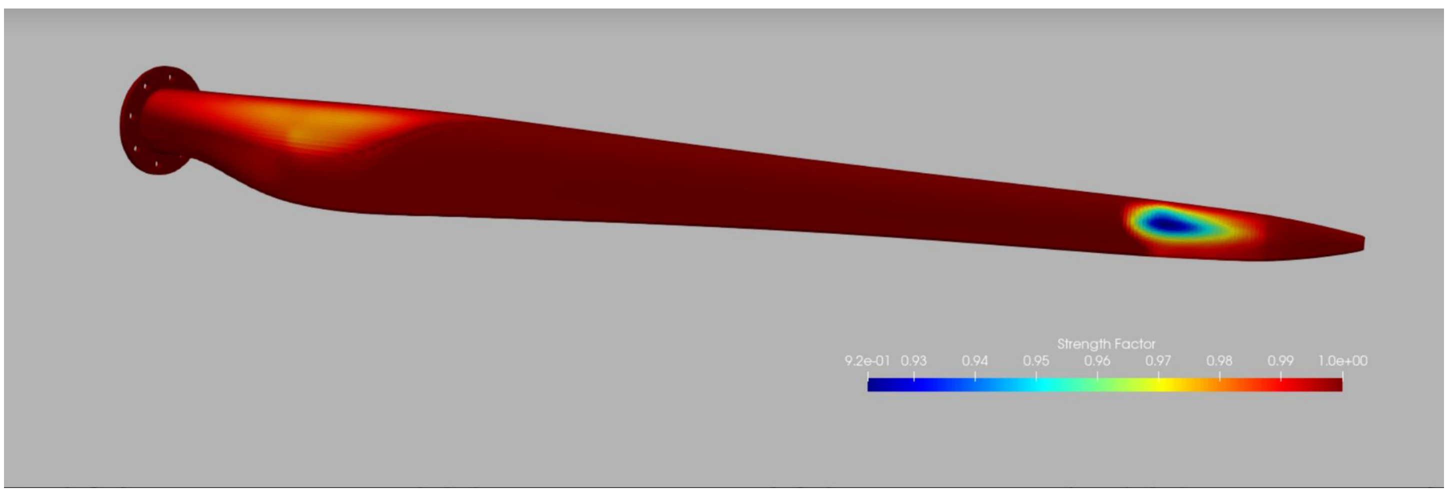




Blade

```
nelem=34K (Hex,Pr)
nsens=19
nload= 1
CALCULIX
```

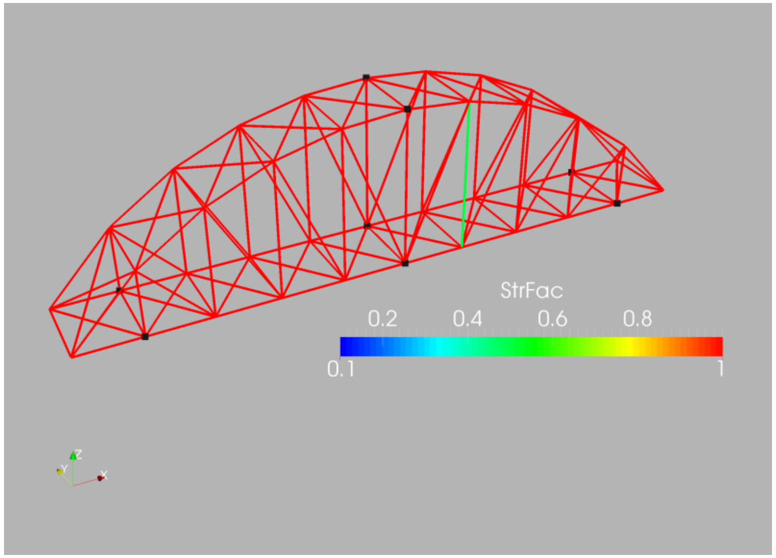


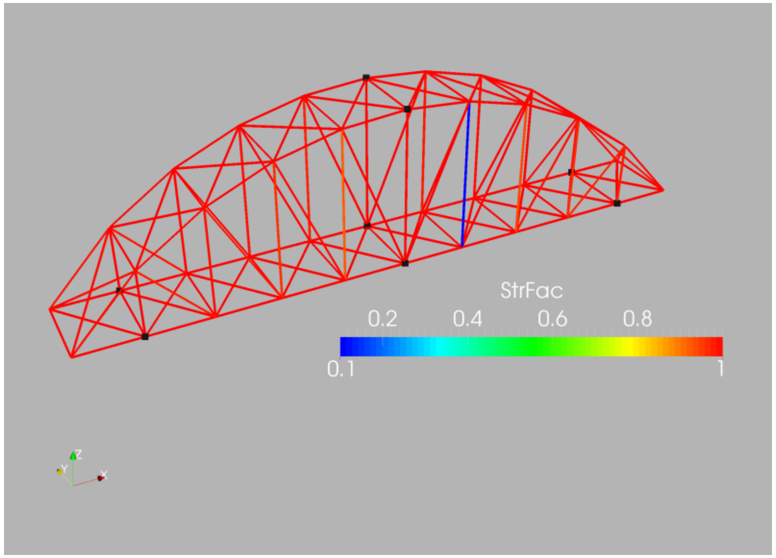


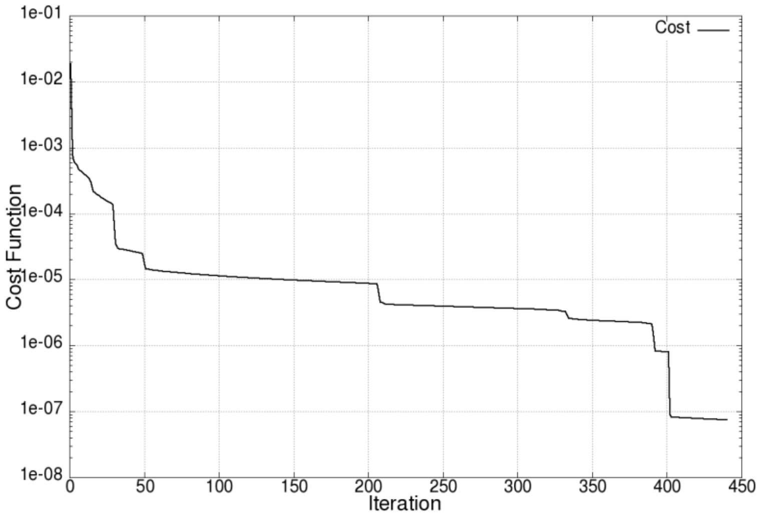
Is The Solution Stable ?

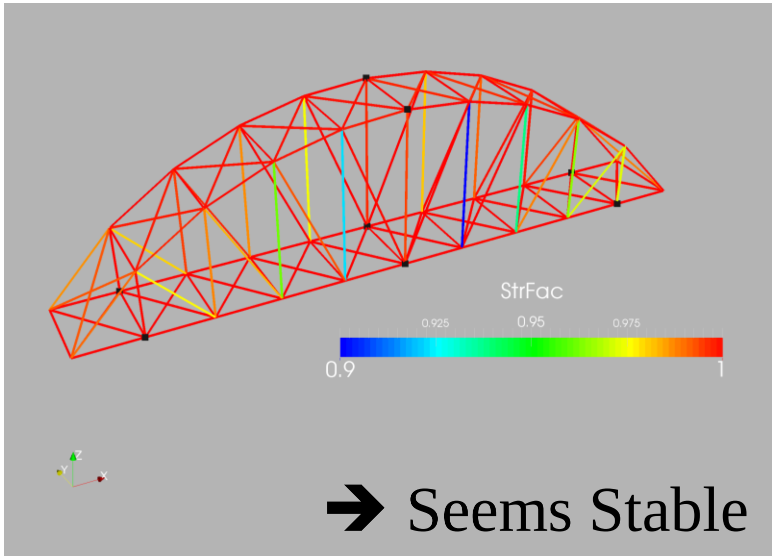
- Start From Close to the Exact Solution
- See If Exact Solution Is Obtained
- Here: Start With 0.2/0.5

Bridge: Start From 0.5

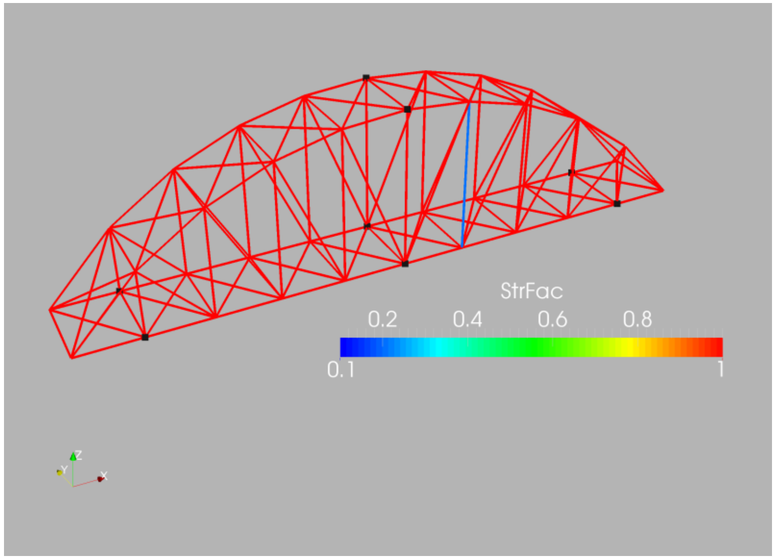


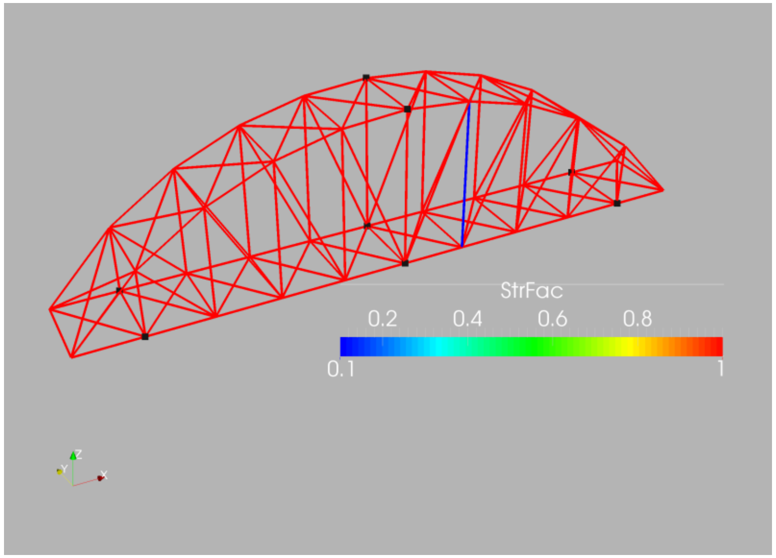


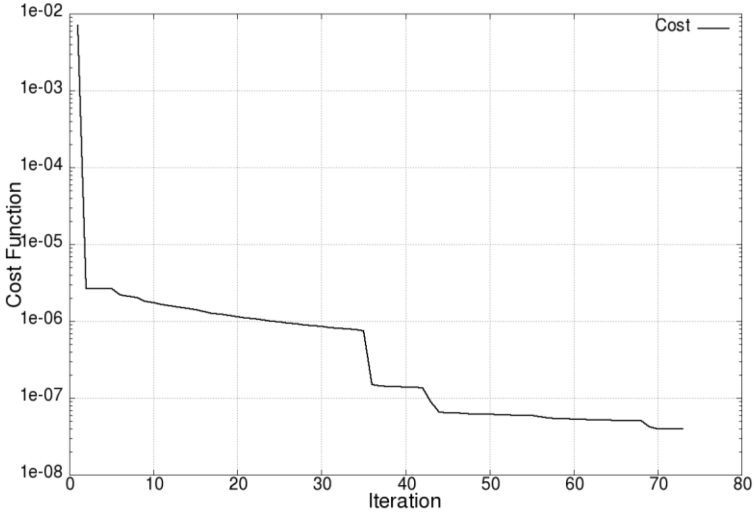


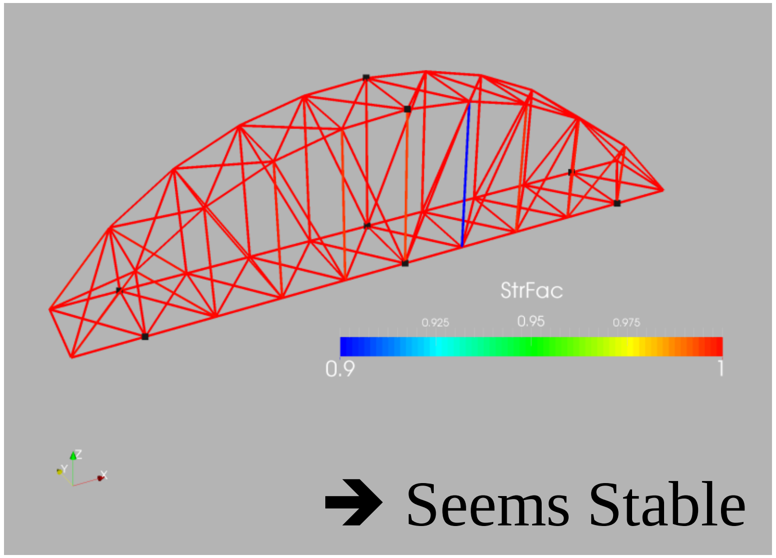


Bridge: Start From 0.2

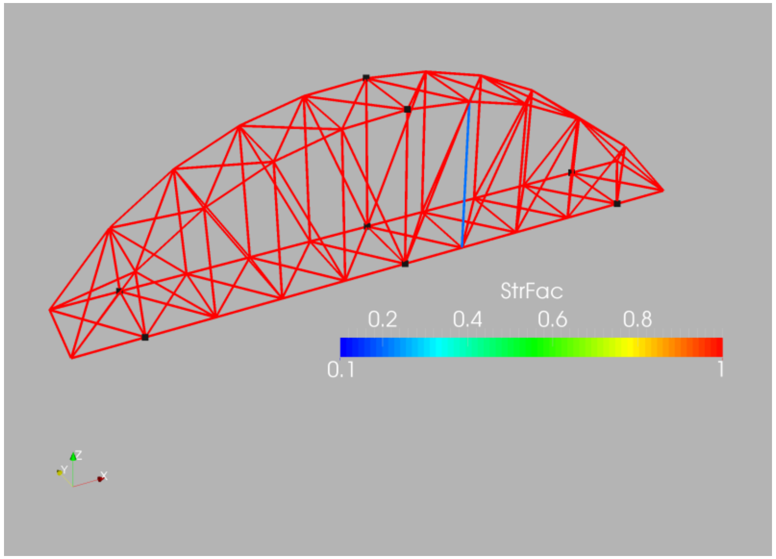


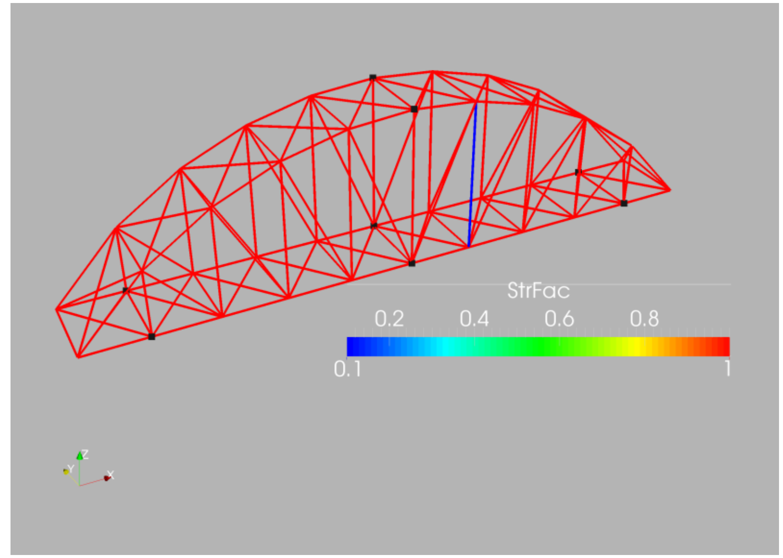


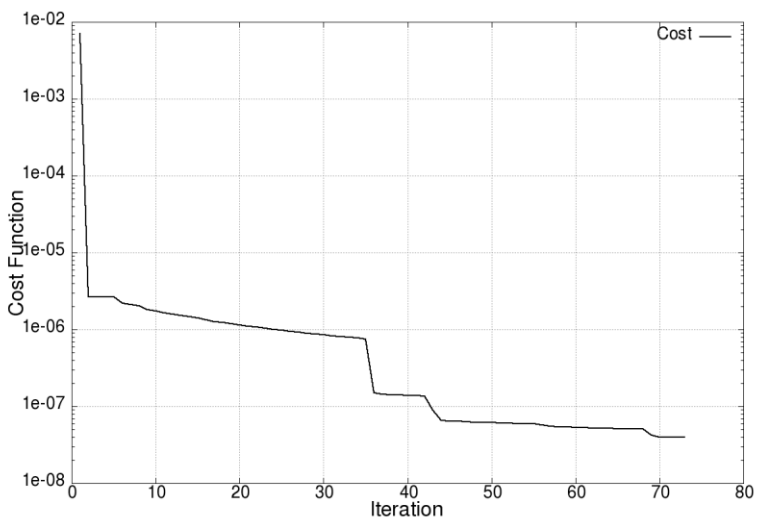


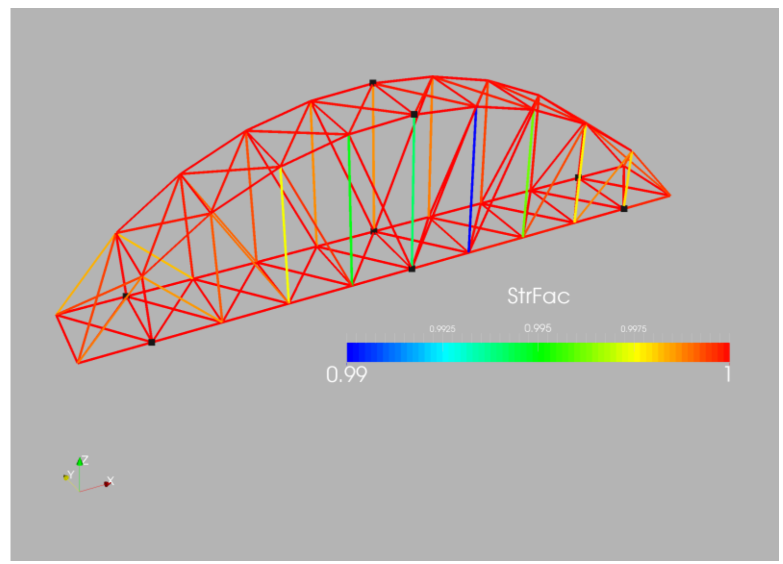


Bridge: Start From 0.2









EXTENSION TO TRANSIENT PROBLEMS

Find $\alpha(\mathbf{x})$ Such That

$$I(\mathbf{u}_{1,..,n},\alpha) = \frac{1}{2}\int_0^T \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{md}(\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d}\cdot\mathbf{u}_i)^2 dt$$

$$+\frac{1}{2}\int_0^T \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{ms}(\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s}\mathbf{s}_i)^2 dt \rightarrow min$$

Subject To:

$$\mathbf{M}\cdot\ddot{\mathbf{u}}_i + \mathbf{C}\cdot\dot{\mathbf{u}}_i + \mathbf{K}\cdot\mathbf{u}_i = \mathbf{f}_i \quad , \quad i=1,n \quad ,$$

$$\mathbf{u}_i(t=0) = \mathbf{u}_i^0 \quad , \quad i=1,n \quad ,$$

$$\dot{\mathbf{u}}_i(t=0) = \mathbf{v}_i^0 \quad , \quad i=1,n \quad ,$$

And As Before:

$$\mathbf{K} = \sum_{el} \alpha_{el}\mathbf{K}_{el}$$

EXTEND TO LAGRANGIAN FUNCTIONAL

$$L(\mathbf{u}_{1,..,n}, \alpha, \tilde{\mathbf{u}}_{1,..,n}) = I(\mathbf{u}_{1,..,n}, \alpha)$$

$$+ \int_0^T \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t (\mathbf{M} \cdot \ddot{\mathbf{u}}_i + \mathbf{C} \cdot \dot{\mathbf{u}}_i + \mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i) dt$$

- $\tilde{\mathbf{u}}$: Adjoint Variables

Variation of the Lagrangian WRT Each Component

$$\frac{dL}{d\tilde{\mathbf{u}}_i} = \mathbf{M} \cdot \ddot{\mathbf{u}}_i + \mathbf{C} \cdot \dot{\mathbf{u}}_i + \mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i = 0$$

$$\frac{dL}{d\mathbf{u}_i} = \sum_{j=1}^{m} w_{ij}^{md} \mathbf{I}_{ij}^{d} \cdot (\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)$$

$$+ \sum_{j=1}^{m} w_{ij}^{ms} \mathbf{J}_{ij}^{s} (\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s} \mathbf{s}_i) + \mathbf{M}^t \cdot \ddot{\tilde{\mathbf{u}}}_i - \mathbf{C}^t \cdot \dot{\tilde{\mathbf{u}}}_i + \mathbf{K}^t \cdot \tilde{\mathbf{u}}_i = 0$$

$$\frac{dL}{d\alpha_e} = \int_0^T \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \cdot \frac{d\mathbf{K}}{d\alpha_e} \cdot \mathbf{u}_i dt = \int_0^T \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \cdot \mathbf{K}^e \cdot \mathbf{u}_i dt \quad ,$$

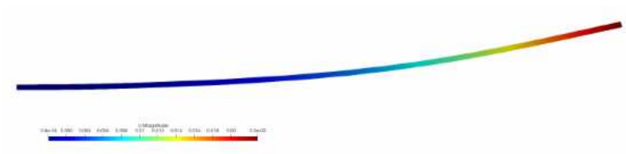
- $\mathbf{J}_{ij}^{s}$: Relationship Displacements/Strains

OPTIMIZATION CYCLE

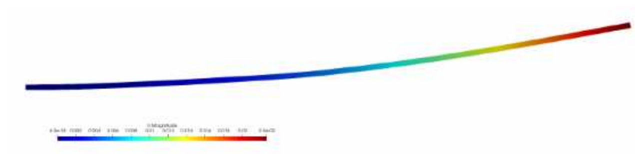
- For Each Force/Measurement Pair i:
- With Current α: Forward Solution, Obtaining Displacements $\rightarrow \mathbf{u}_i(t)$
- With Current α, $\mathbf{u}_i(t)$, $\mathbf{u}_{ij}^{md}, \mathbf{s}_{ij}^{md}$: Backwards Solution, Obtaining Adjoints $\rightarrow \tilde{\mathbf{u}}_i(t)$
- With $\mathbf{u}_i(t), \tilde{\mathbf{u}}_i(t)$: Obtain Gradients $\rightarrow I^i_{,\alpha} = L^i_{,\alpha}$
- Sum up the gradients $\rightarrow I_{,\alpha} = \sum_{i=1}^{n} I^i_{,\alpha}$
- If Necessary: Smooth Gradients $\rightarrow I^{smoo}_{,\alpha}$
- Update $\alpha_{new} = \alpha_{old} - \gamma I^{smoo}_{,\alpha}$.
- γ: Stepsize

EXAMPLE: BEAM (1)

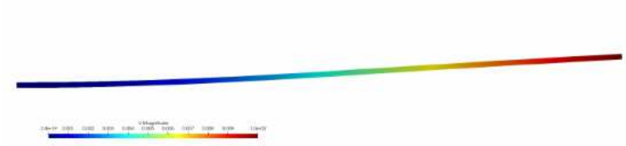
- Beam
- 10 Elements
- Displacement Sensors at Each Point
- CALCULIX



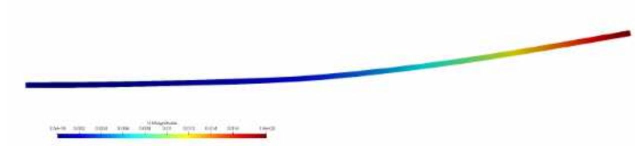
(a) Undamaged. $t = 0.25$ seconds.



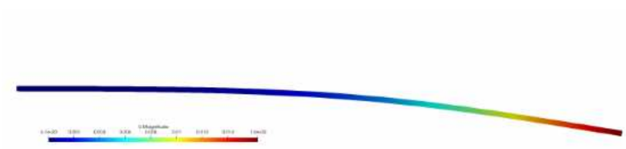
(b) Damaged. $t = 0.25$ seconds.



(c) Undamaged. $t = 0.50$ seconds.

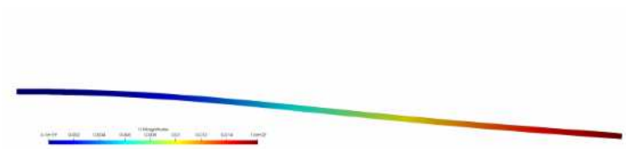


(d) Damaged. $t = 0.50$ seconds.

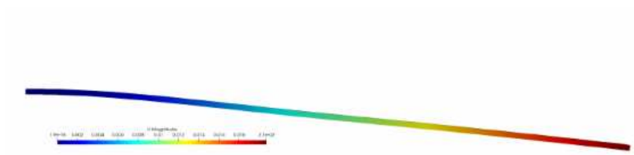


(e) Undamaged. $t = 0.75$ seconds.

(f) Damaged. $t = 0.75$ seconds.

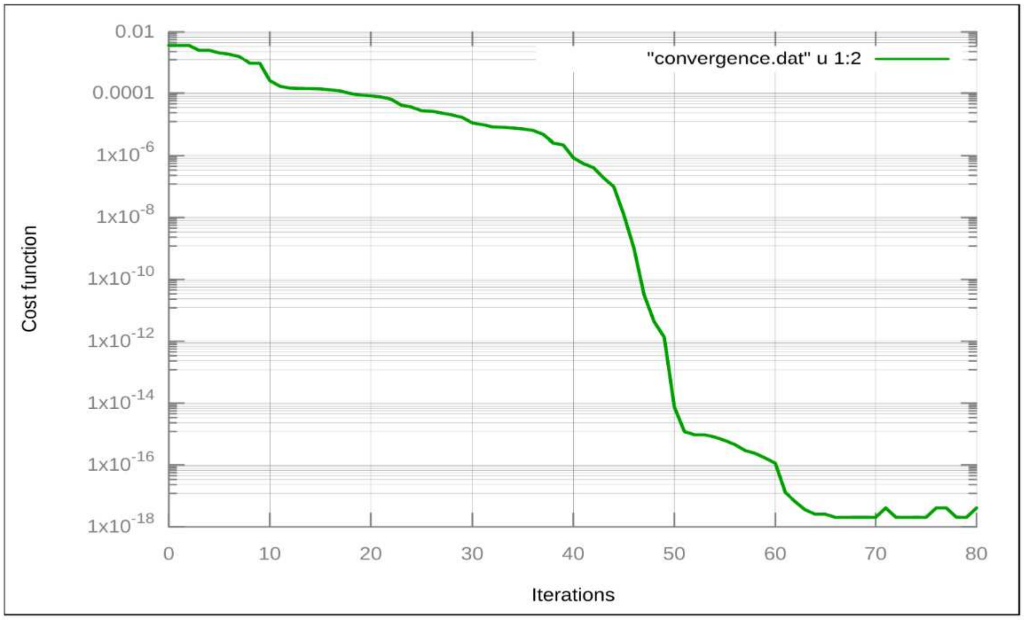


(g) Undamaged. $t = 1.0$ seconds.



(h) Damaged. $t = 1.0$ seconds.

EXAMPLE: BEAM (2)



Convergence of Cost Function

MOTIVATION: PARAMETER ESTIMATION

- In Many Cases
  - Material Behaviour Unknown
  - Material Parameters Unknown
- Modus Operandi:
  - Load Structure
  - Measure Displacements/Strains
  - For Each Possible Material Model
    - Obtain Best Parameter Set
  - Keep 'Best Fitting' Model

ASSUMPTIONS

- Monitoring Via Loads and Measurements:
  - n (Standard) Loads $\mathbf{f}^i, i = 1, n$ Given
  - n Displacements/Strains Measured at m Locations
- ‘Uniform Materials’
- Material Parameters Described by $\boldsymbol{\beta}$
- Deformations and Strains Well Described by FEM:

$$\mathbf{K} \cdot \mathbf{u}_i = \mathbf{f}_i \quad , \quad i = 1, n \quad , \quad \mathbf{K} = \sum_{el} \mathbf{K}_{el}(\boldsymbol{\beta})$$

FORMULATE AS OPTIMIZATION PROBLEM

Find $\boldsymbol{\beta}$ Such That

$$I(\mathbf{u}_{1,..,n}, \boldsymbol{\beta}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{md}(\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{ms}(\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s}\mathbf{s}_i)^2 \rightarrow min$$

Subject To:

$$\mathbf{K} \cdot \mathbf{u}_i = \mathbf{f}_i \quad , \quad i = 1, n \quad , \quad \mathbf{K} = \sum_{el} \mathbf{K}_{el}(\boldsymbol{\beta})$$

EXTEND TO LAGRANGIAN FUNCTIONAL

$$L(\mathbf{u}_{1,..,n}, \boldsymbol{\beta}, \tilde{\mathbf{u}}_{1,..,n}) = I(\mathbf{u}_{1,..,n}, \boldsymbol{\beta}) + \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \cdot (\mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i)$$

- $\tilde{\mathbf{u}}$: Adjoint Variables

Variation of the Lagrangian WRT Each Component

$$\frac{dL}{d\tilde{\mathbf{u}}_i} = \mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i = 0$$

$$\frac{dL}{d\mathbf{u}_i} = \sum_{j=1}^{m} w_{ij}^{md} \mathbf{I}_{ij}^{d} \cdot (\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)$$

$$+ \sum_{j=1}^{m} w_{ij}^{ms} \mathbf{J}_{ij}^{s} \cdot (\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s} \mathbf{s}_i) + \mathbf{K}^t \tilde{\mathbf{u}}_i = 0$$

$$\frac{dL}{d\boldsymbol{\beta}_i} = \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \frac{\cdot d\mathbf{K}}{d\boldsymbol{\beta}} \cdot \mathbf{u}_i = \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \frac{\cdot d\mathbf{K}^e}{d\boldsymbol{\beta}} \cdot \mathbf{u}_i \quad ,$$

- $\mathbf{J}_{ij}^{s}$: Relationship Displacements/Strains

USE OF ADJOINT SOLVERS

- Consequences
  - Gradient of L wrt $\boldsymbol{\beta}$: n Forward/Adjoint Solves;
  - **Independent** of Number of Variables of $\boldsymbol{\beta}$ (!);
  - Once Forward/Adjoint Solves Done: Cost of Gradient of Each $\boldsymbol{\beta}$ Is $O(N_e)$;

  $\Rightarrow$ Can Use Detailed FEM Models $\rightarrow$ Detailed Digital Twin
  - 'Hi-Fi Digital Twin'
  - Based on Algebraic Equations (FEMs), Not PDEs $\Rightarrow$ General
- Most Structural Problems: $\mathbf{K} =^t \mathbf{K} \Rightarrow$
  - Direct Solvers: Cost of Adjoint Negligible
  - Iterative Solver: Preconditioner Can Be Re-Utilized

OPTIMIZATION CYCLE

- For Each Force/Measurement Pair i:
- With Current $\boldsymbol{\beta}$: Obtain Displacements $\rightarrow \mathbf{u}_i$
- With Current $\boldsymbol{\beta}$, $\mathbf{u}_i$, $\mathbf{u}_{ij}^{md}, \mathbf{s}_{ij}^{md}$: Obtain Adjoints $\rightarrow \tilde{\mathbf{u}}_i$
- With $\mathbf{u}_i, \tilde{\mathbf{u}}_i$: Obtain Gradients $\rightarrow I^i_{,\boldsymbol{\beta}} = L^i_{,\boldsymbol{\beta}}$
- Sum up the gradients $\rightarrow I_{,\boldsymbol{\beta}} = \sum_{i=1}^{n} I^i_{,\boldsymbol{\beta}}$
- If Necessary: Smooth Gradients (Usually Not Needed) $\rightarrow I^{smoo}_{,\boldsymbol{\beta}}$
- Update $\boldsymbol{\beta}_{new} = \boldsymbol{\beta}_{old} - \gamma I^{smoo}_{,\boldsymbol{\beta}}$.
- γ: Stepsize

MOTIVATION: FORCE ESTIMATION

- In Many Cases Forces Unknown
- Modus Operandi:
    - Know Structure/Material Parameters
    - Measure Displacements/Strains
    - Recover Forces

ASSUMPTIONS

- Structure Known
    - Materials
    - Material Parameters
- Deformations and Strains Well Described by FEM:

$$\mathbf{K} \cdot \mathbf{u} = \mathbf{f} \quad , \quad \mathbf{K} = \sum_{el} \mathbf{K}_{el}$$

FORMULATE AS OPTIMIZATION PROBLEM

Find **f** Such That

$$I(\mathbf{u}, \mathbf{f}) = \frac{1}{2} \sum_{j=1}^{m} w_{ij}^{md} (\mathbf{u}_j^{md} - \mathbf{I}_j^d \cdot \mathbf{u})^2$$

$$+ \frac{1}{2} \sum_{j=1}^{m} w_j^{ms} (\mathbf{s}_j^{ms} - \mathbf{I}_j^s \mathbf{s})^2 \rightarrow min$$

Subject To:

$$\mathbf{K} \cdot \mathbf{u} = \mathbf{f}$$

EXTEND TO LAGRANGIAN FUNCTIONAL

$$L(\mathbf{u}, \mathbf{f}, \tilde{\mathbf{u}}) = I(\mathbf{u}, \mathbf{f}) + \tilde{\mathbf{u}}^t \cdot (\mathbf{K} \cdot \mathbf{u} - \mathbf{f})$$

- $\tilde{\mathbf{u}}$: Adjoint Variables

Variation of the Lagrangian WRT Each Component

$$\frac{dL}{d\tilde{\mathbf{u}}} = \mathbf{K} \cdot \mathbf{u} - \mathbf{f} = 0$$

$$\frac{dL}{d\mathbf{u}} = \sum_{j=1}^{m} w_j^{md} \mathbf{I}_j^d (\mathbf{u}_j^{md} - \mathbf{I}_j^d \mathbf{u})$$

$$+ \sum_{j=1}^{m} w_j^{ms} \mathbf{J}_j^s (\mathbf{s}_j^{ms} - \mathbf{I}_j^s \mathbf{s}) + \mathbf{K}^t \tilde{\mathbf{u}} = 0$$

$$\frac{dL}{d\mathbf{f}} = -\tilde{\mathbf{u}}^t \mathbf{M}_v$$

USE OF ADJOINT SOLVERS

- Consequences
  - Gradient of L wrt $\mathbf{f}$: One Forward/Adjoint Solve;
  - **Independent** of Number of Variables of $\mathbf{f}$ (!);
  - Once Forward/Adjoint Solves Done: Cost of Gradient of Each $\mathbf{f}$ Is $O(1)$;

  $\Rightarrow$ Can Use Detailed FEM Models $\rightarrow$ Detailed Digital Twin
  - 'Hi-Fi Digital Twin'
  - Based on Algebraic Equations (FEMs), Not PDEs $\Rightarrow$ General

- Most Structural Problems: $\mathbf{K} =^t \mathbf{K} \Rightarrow$
  - Direct Solvers: Cost of Adjoint Negligible
  - Iterative Solver: Preconditioner Can Be Re-Utilized

OPTIMIZATION CYCLE

- With Current $\mathbf{f}$: Obtain Displacements $\rightarrow \mathbf{u}$
- With Current $\mathbf{f}$, $\mathbf{u}$, $\mathbf{u}_j^{md}, \mathbf{s}_j^{md}$: Obtain Adjoints $\rightarrow \tilde{\mathbf{u}}$
- With $\mathbf{u}, \tilde{\mathbf{u}}$: Obtain Gradients $\rightarrow I_{,\mathbf{f}} = L_{,\mathbf{f}}$
- If Necessary: Smooth Gradients (Usually Not Needed) $\rightarrow I_{,\mathbf{f}}^{smoo}$
- Update $\mathbf{f}_{new} = \mathbf{f}_{old} - \gamma I_{,\mathbf{f}}^{smoo}$.
- γ: Stepsize

MOTIVATION: TEMPERATURE ESTIMATION

- In Many Cases
  - Temperature Distribution Unknown
    (Solar Irradiation, Shading, Wind, ...)
  - Material Parameters Known

- Modus Operandi:
  - Load Structure
  - Measure Displacements/Strains
  - Recover Temperature

THERMAL EXPANSION/CONTRACTION (1)

- 1-D Rod:
  - Length L
  - Temperature Increment $\Delta T \Rightarrow$ Expansion ΔL:

$$\Delta L = \alpha L \Delta T$$

- α: Coefficient of Thermal Expansion $\Rightarrow$ Strain ϵ:

$$\epsilon = \alpha \Delta T$$

For Isotropic Thermal Strain:

$$\sigma^{ij} = -\frac{\alpha E}{1-2\nu} \Delta T \delta^{ij}$$

- E: Young's Modulus
- ν: Poisson Coefficient

THERMAL EXPANSION/CONTRACTION (2)

Conservation of Momentum (Forces):

$$\sigma^{ij}_{,j} = f$$

Internal Forces f_i^k Due to Thermal Strain:

$$f^k_{int} = -\left(\frac{\alpha E}{1-2\nu}\Delta T\right)_{,k}$$

THERMAL EXPANSION/CONTRACTION (3)

FEM/Integration by Parts:

$$\int N^i f_{int}^k d\Omega = \int N^i_{,k} \frac{\alpha E}{1-2\nu} N^j \Delta \hat{T}_j d\Omega$$

- N^i: Shape Function of Node/DOF i

For Linear Elements:

$$\hat{f}^i_k = \sum_{els} N^i_{,k} \frac{\alpha E}{1-2\nu} Vol_{el} \Delta T^{ave}_{el} \quad , \quad T^{ave}_{el} = \frac{1}{nn} \sum_{j=1}^{nn} \Delta \hat{T}_j$$

ASSUMPTIONS

- Monitoring Via Loads and Measurements:
  - n (Standard) Loads $\mathbf{f}^i, i = 1, n$ Given
  - n Displacements/Strains Measured at m Locations
- 'Uniform Materials'
- Deformations and Strains Well Described by FEM:

$$\mathbf{K} \cdot \mathbf{u}_i = \mathbf{f}_i \quad , \quad i = 1, n \quad , \quad \mathbf{K} = \sum_{el} \mathbf{K}_{el}$$

FORMULATE AS OPTIMIZATION PROBLEM

Find $\mathbf{\Delta T}$ Such That

$$I(\mathbf{u}_{1,..,n}, \mathbf{\Delta T}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{md}(\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^{d} \cdot \mathbf{u}_i)^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{ms}(\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^{s}\mathbf{s}_i)^2 \rightarrow min$$

Subject To:

$$\mathbf{K} \cdot \mathbf{u}_i = \mathbf{f}_i + \mathbf{f}_{\mathbf{\Delta T}} \quad , \quad i = 1, n$$

EXTEND TO LAGRANGIAN FUNCTIONAL

$$L(\mathbf{u}_{1,..,n}, \Delta\mathbf{T}, \tilde{\mathbf{u}}_{1,..,n}) = I(\mathbf{u}_{1,..,n}, \Delta\mathbf{T})$$

$$+\sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \cdot (\mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i - \mathbf{f}_{\Delta\mathbf{T}})$$

- $\tilde{\mathbf{u}}$: Adjoint Variables

Variation of the Lagrangian WRT Each Component

$$\frac{dL}{d\tilde{\mathbf{u}}_i} = \mathbf{K} \cdot \mathbf{u}_i - \mathbf{f}_i - \mathbf{f}_{\Delta\mathbf{T}} = 0$$

$$\frac{dL}{d\mathbf{u}_i} = \sum_{j=1}^{m} w_{ij}^{md} \mathbf{I}_{ij}^d \cdot (\mathbf{u}_{ij}^{md} - \mathbf{I}_{ij}^d \cdot \mathbf{u}_i)$$

$$+\sum_{j=1}^{m} w_{ij}^{ms} \mathbf{J}_{ij}^s (\mathbf{s}_{ij}^{ms} - \mathbf{I}_{ij}^s \mathbf{s}_i) + \mathbf{K}^t \tilde{\mathbf{u}}_i = 0$$

$$\frac{dL}{d\Delta\mathbf{T}} = \sum_{i=1}^{n} \tilde{\mathbf{u}}_i^t \frac{d\mathbf{f}_{\Delta\mathbf{T}}}{d\Delta\mathbf{T}}$$

- $\mathbf{J}_{ij}^s$: Relationship Displacements/Strains

USE OF ADJOINT SOLVERS

- Consequences
  - Gradient of L wrt $\Delta\mathbf{T}$: n Forward/Adjoint Solves;
  - **Independent** of Number of Variables of $\Delta\mathbf{T}$ (!);
  - Once Forward/Adjoint Solves Done: Cost of Gradient of Each ΔT_i Is $O(1)$;

  $\Rightarrow$ Can Use Detailed FEM Models $\rightarrow$ Detailed Digital Twin
  - 'Hi-Fi Digital Twin'
  - Based on Algebraic Equations (FEMs), Not PDEs $\Rightarrow$ General

- Most Structural Problems: $\mathbf{K} =^t \mathbf{K} \Rightarrow$
  - Direct Solvers: Cost of Adjoint Negligible
  - Iterative Solver: Preconditioner Can Be Re-Utilized

OPTIMIZATION CYCLE

- For Each Force/Measurement Pair i:
- With Current $\Delta\mathbf{T}$: Obtain Displacements $\rightarrow \mathbf{u}_i$
- With Current $\Delta\mathbf{T}$, $\mathbf{u}_i$, $\mathbf{u}^{md}_{ij}, \mathbf{s}^{md}_{ij}$: Obtain Adjoints $\rightarrow \tilde{\mathbf{u}}_i$
- With $\mathbf{u}_i, \tilde{\mathbf{u}}_i$: Obtain Gradients $\rightarrow I^i_{,\Delta\mathbf{T}} = L^i_{,\Delta\mathbf{T}}$
- Sum up the gradients $\rightarrow I_{,\Delta\mathbf{T}} = \sum_{i=1}^{n} I^i_{,\Delta\mathbf{T}}$
- If Necessary: Smooth Gradients (Usually Not Needed) $\rightarrow I^{smoo}_{,\Delta\mathbf{T}}$
- Update $\Delta\mathbf{T}new = \Delta\mathbf{T}old - \gamma I^{smoo}_{,\Delta\mathbf{T}}$.
- γ: Stepsize

Plate With Hole (MP1)

- Linear Elastic
- Material Parameters: E, ν
- 4, 6 and 14 Sensors

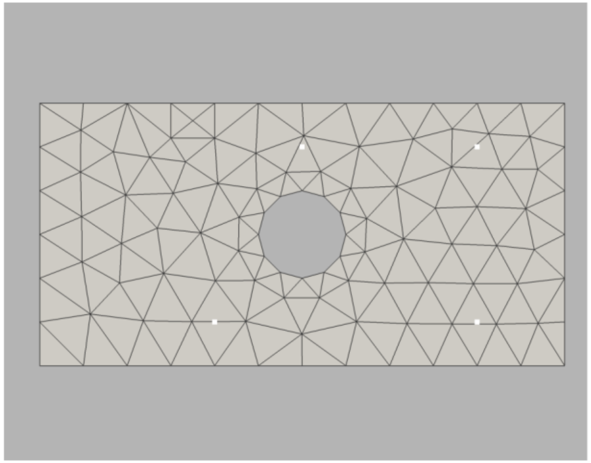
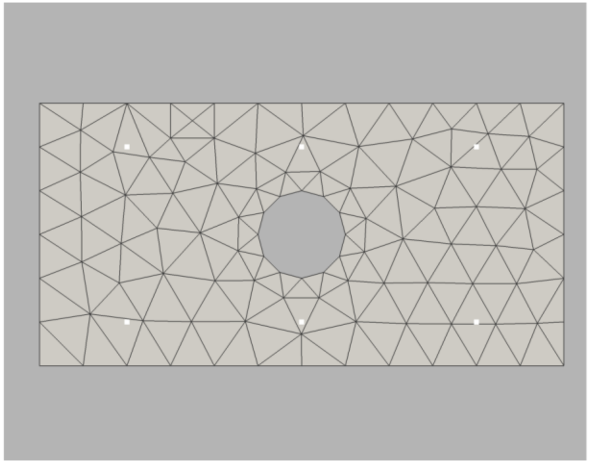
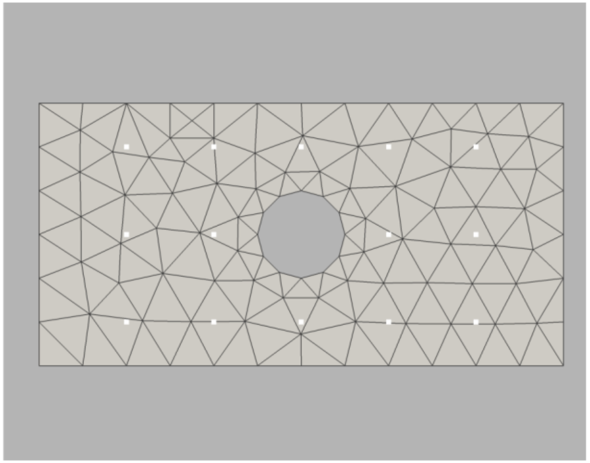




Plate With Hole (MP2)

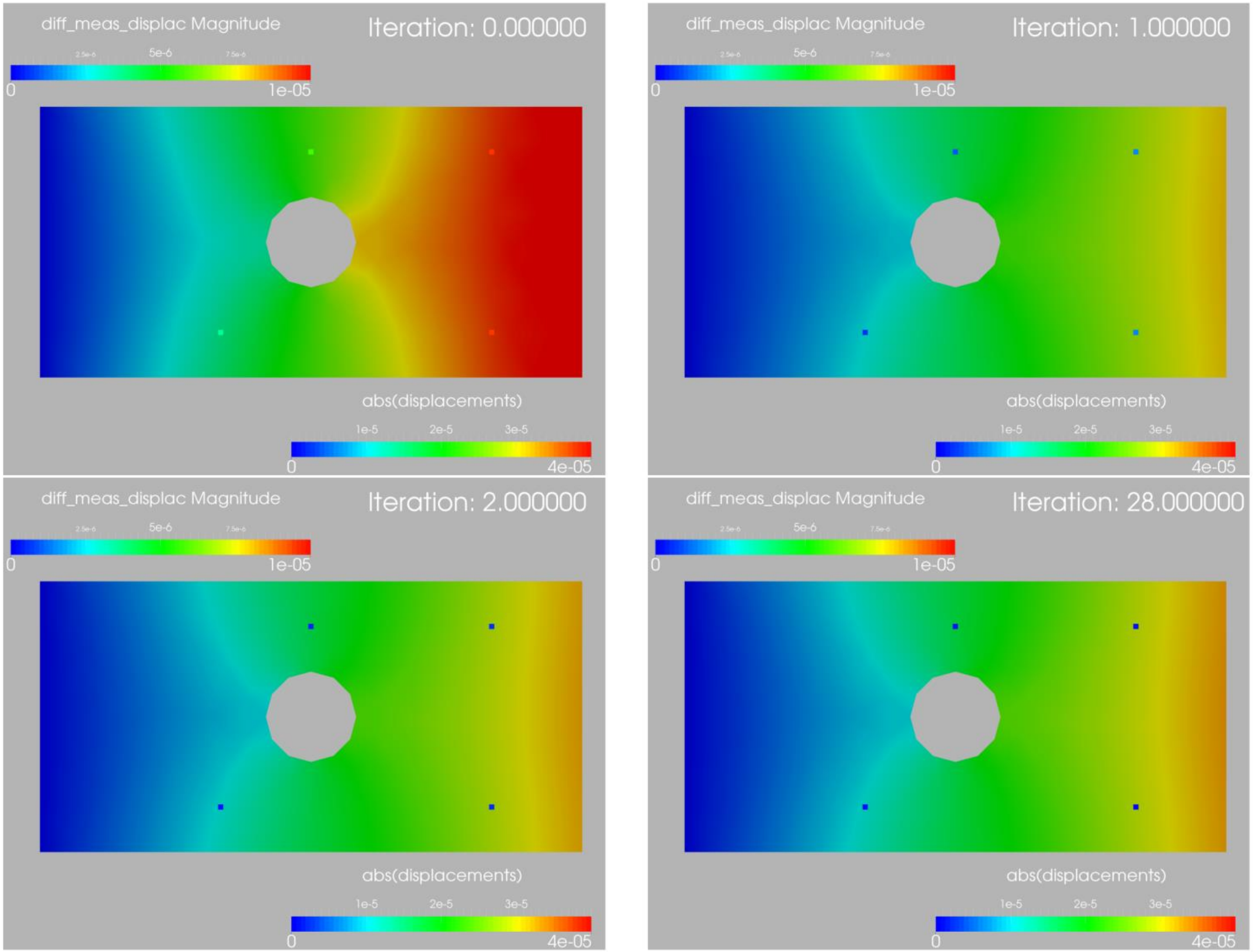
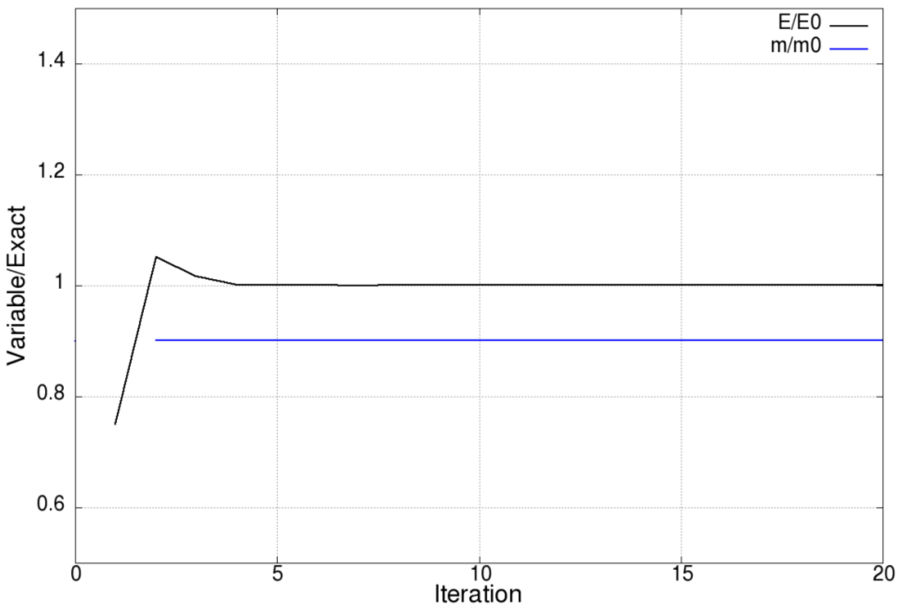


Plate With Hole (MP3a)

4 Sensors



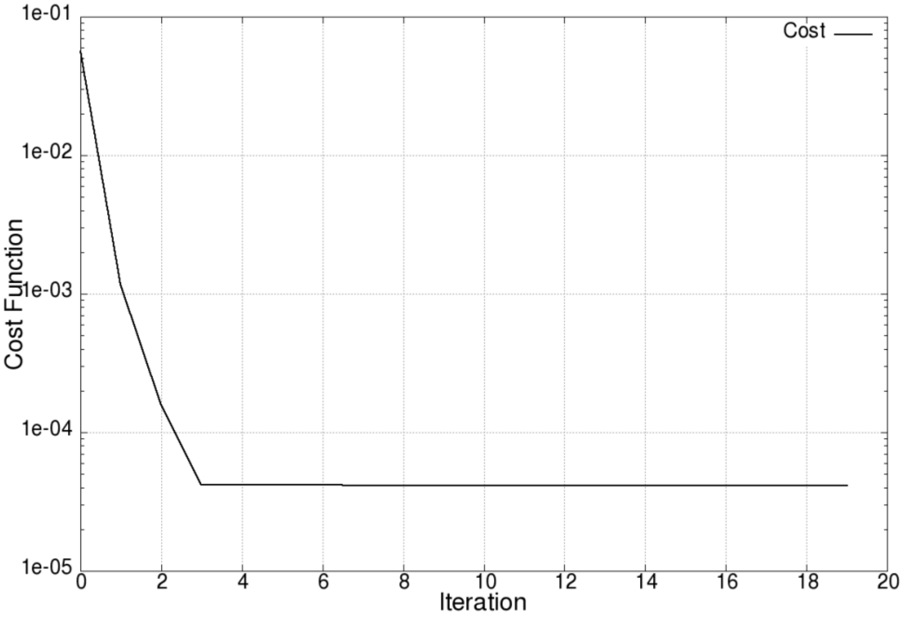
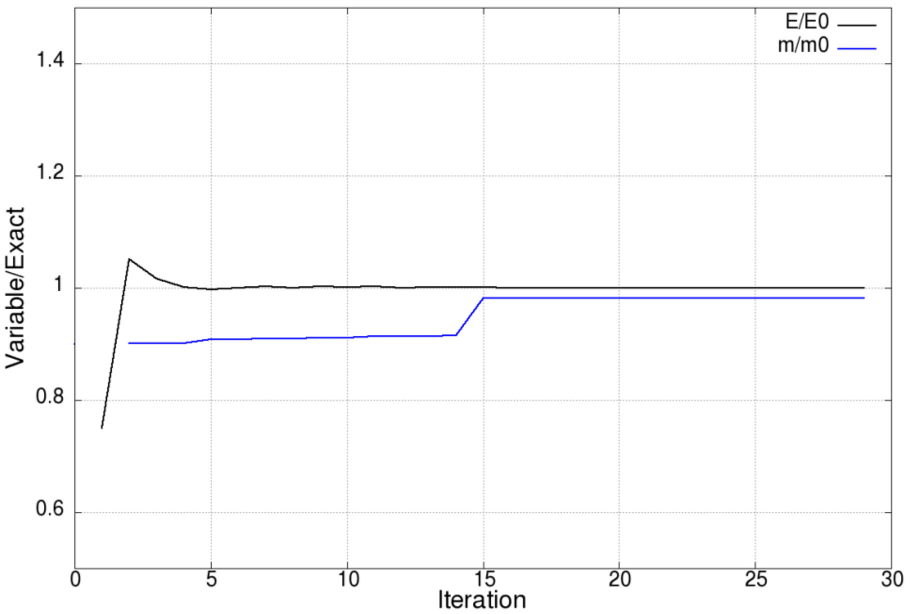


Plate With Hole (MP3b)

6 Sensors



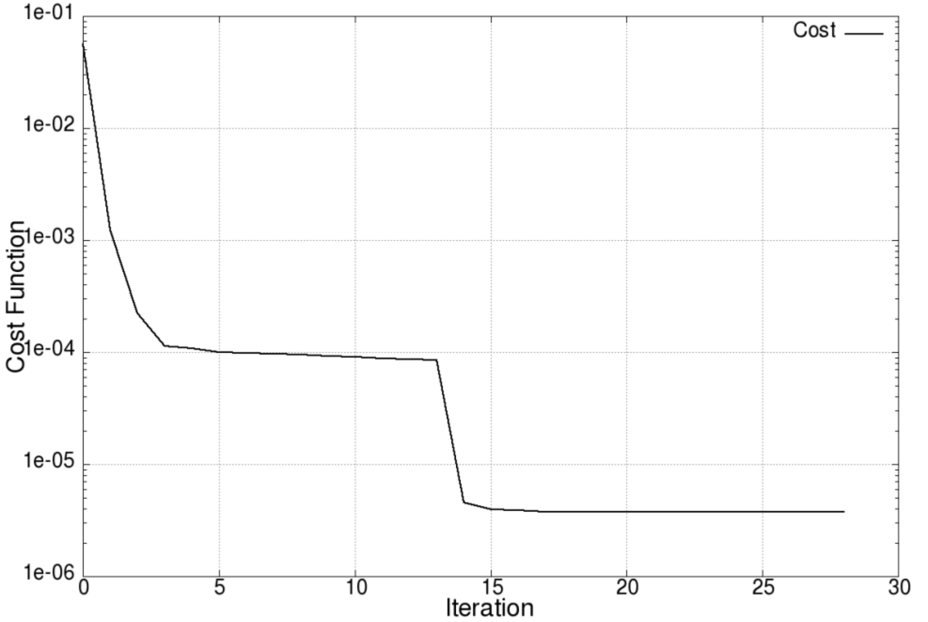
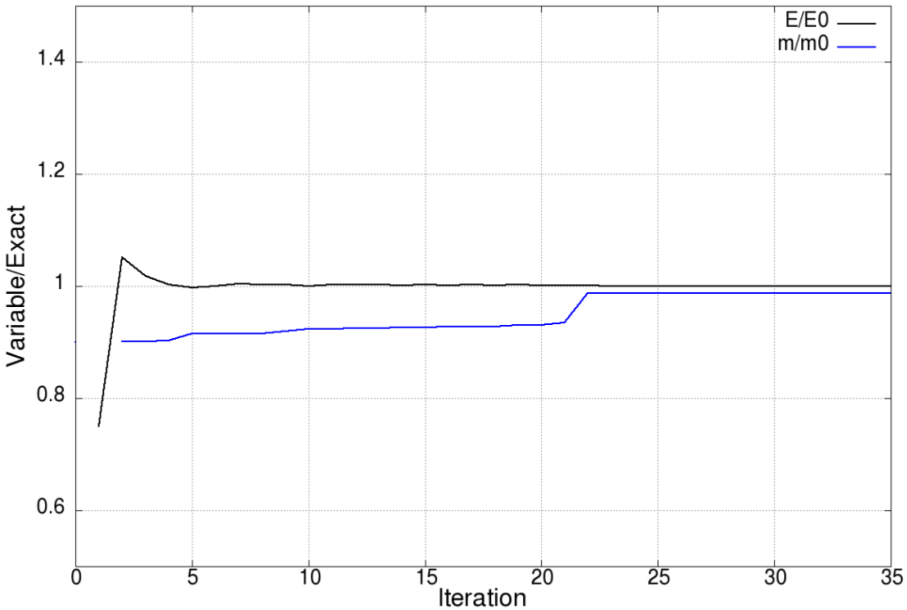
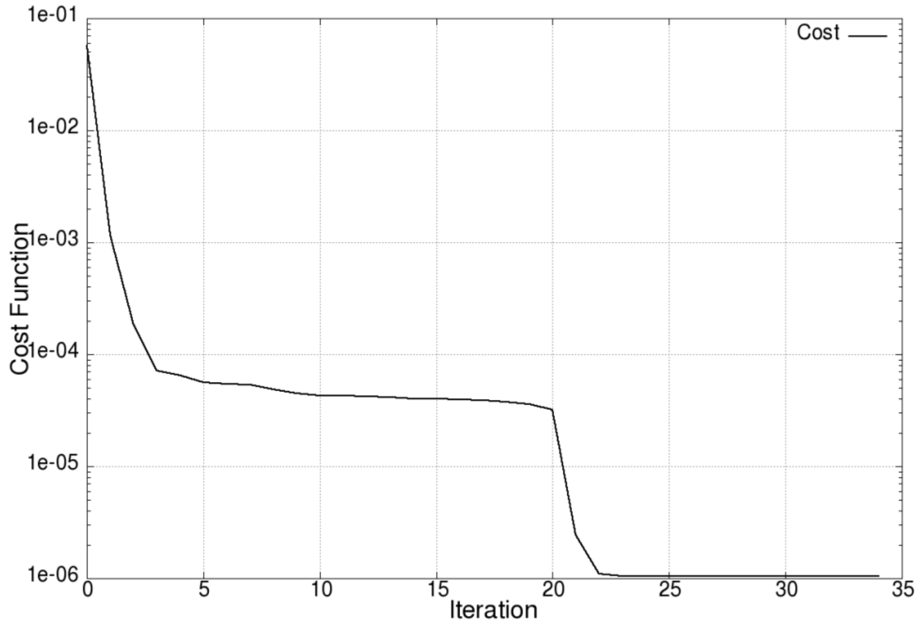


Plate With Hole (MP3c)

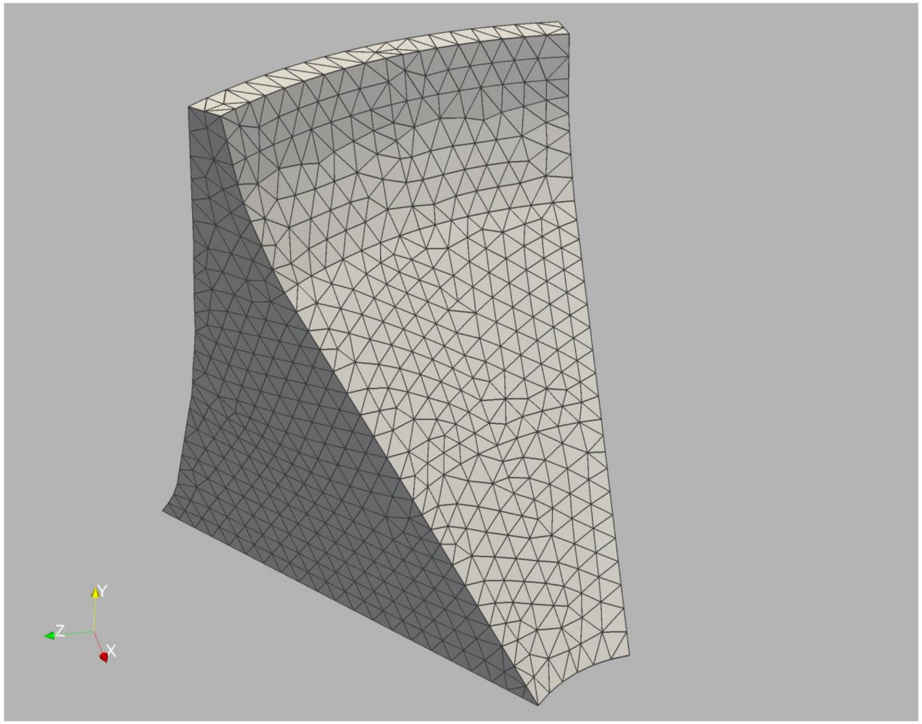
14 Sensors

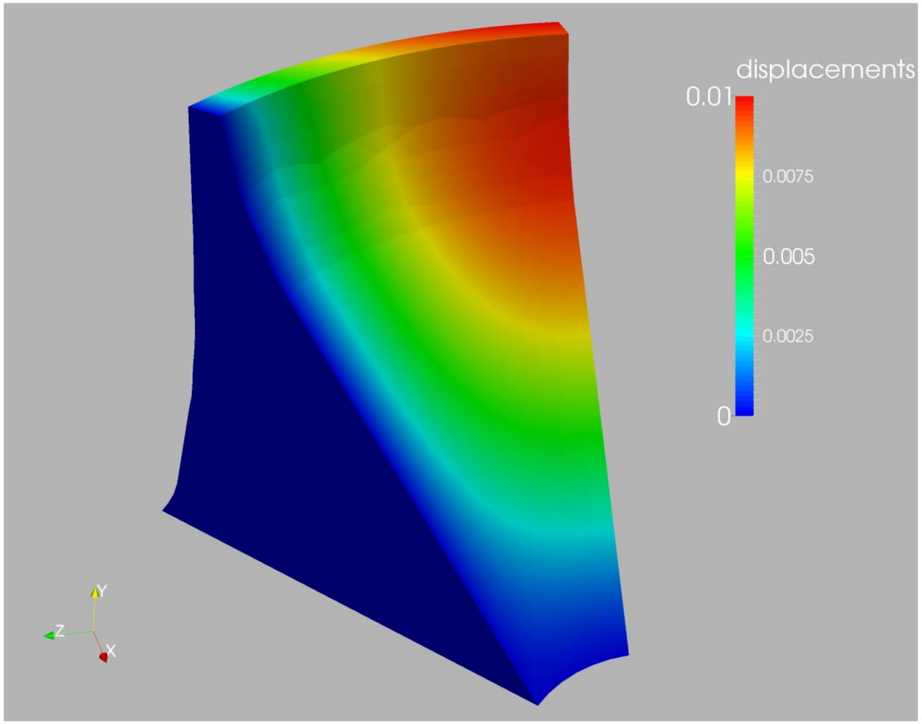




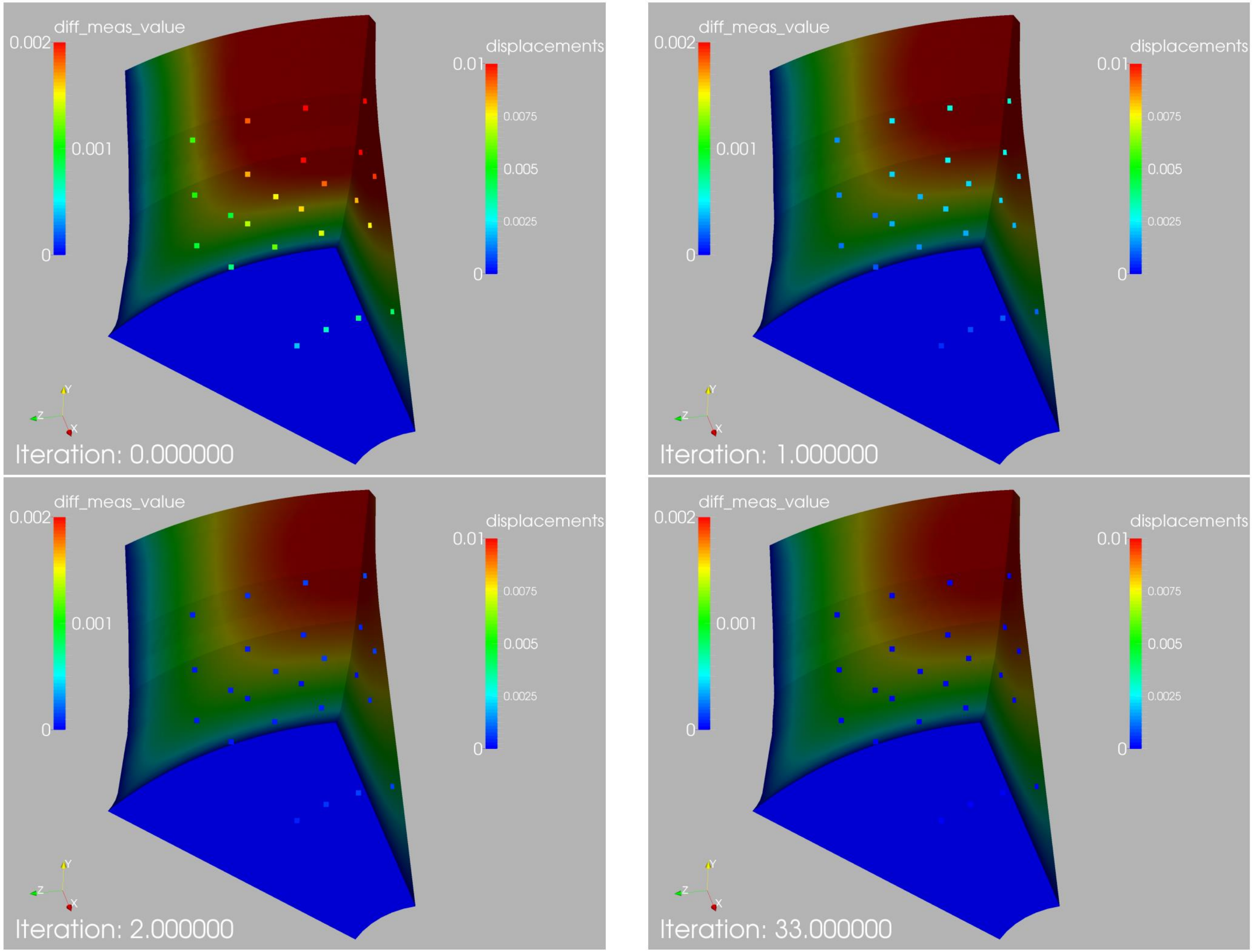
Hoover Dam (MP1)

- Linear Elastic
- Material Parameters: E, ν
- Mesh and Target Displacement



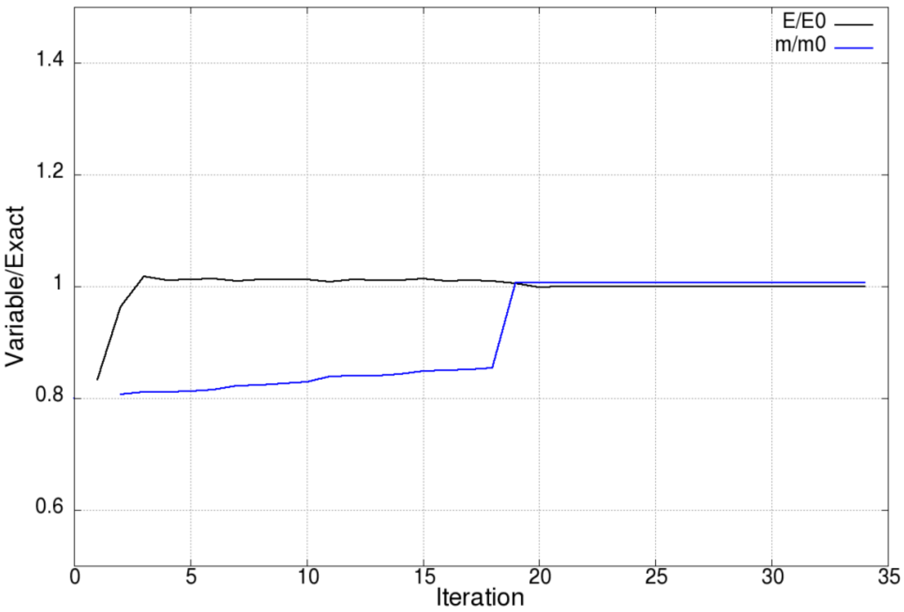


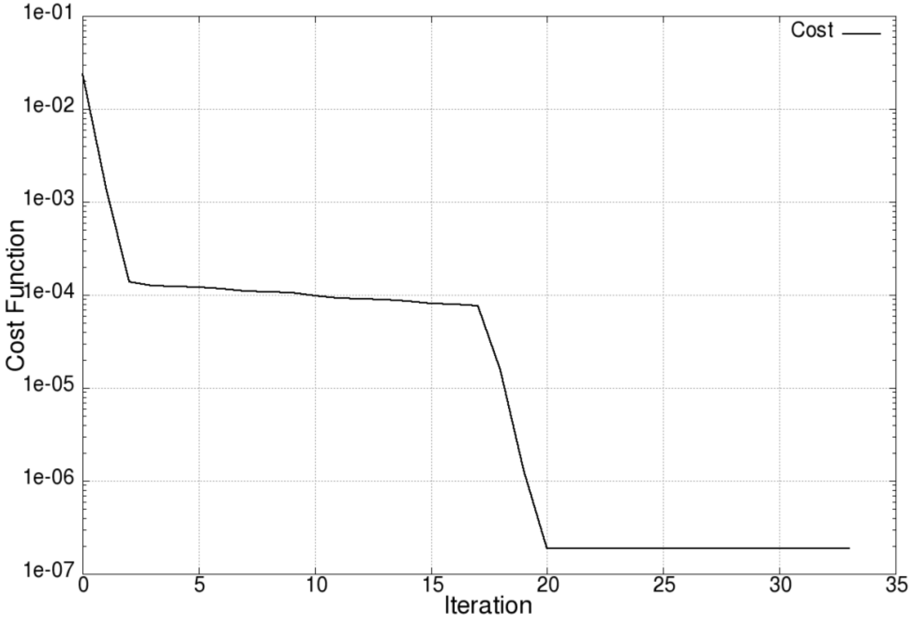
Hoover Dam (MP2)



Hoover Dam (MP3)

- Convergence History



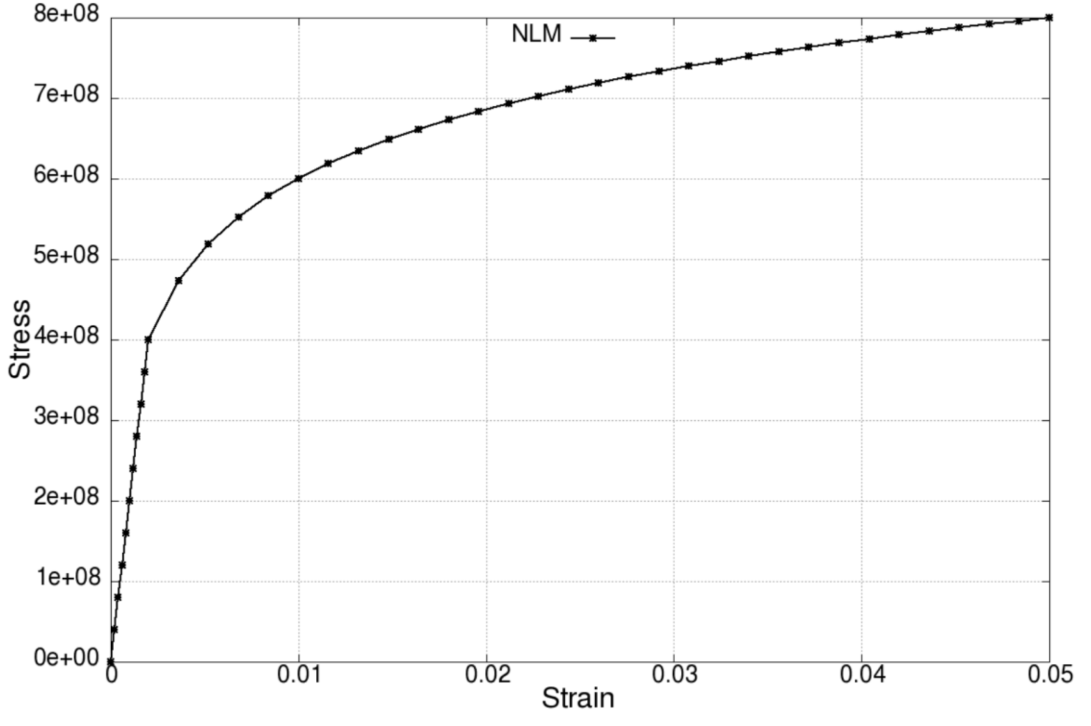


Truss (MP1)

- Nonlinear Material

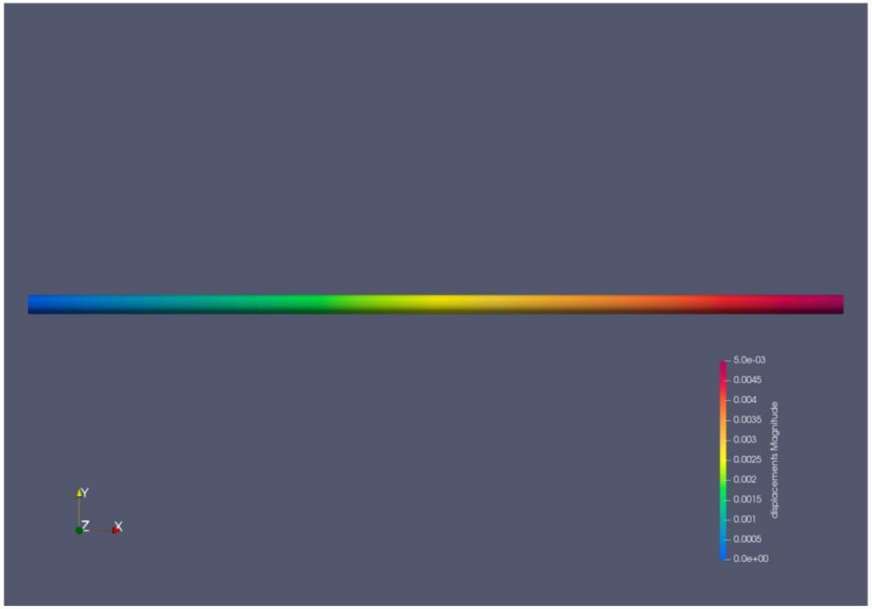
$$\epsilon \leq \epsilon_0 : E = E_0$$

$$\epsilon \geq \epsilon_0 : E_0\epsilon_0 + (E_1\epsilon_1 - E_0\epsilon_0)(\log(\epsilon/\epsilon_0))/(\log(\epsilon_1/\epsilon_0))$$



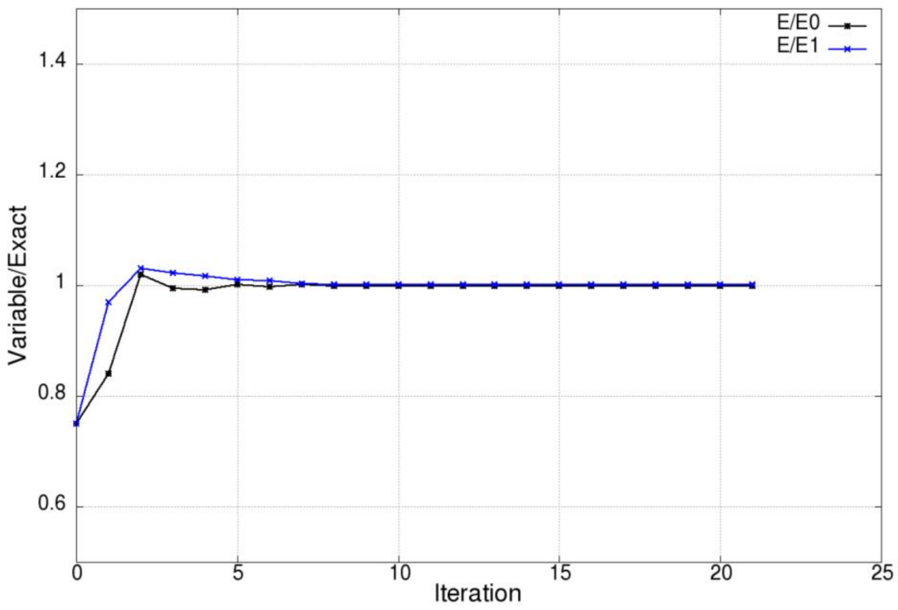
Truss (MP2)

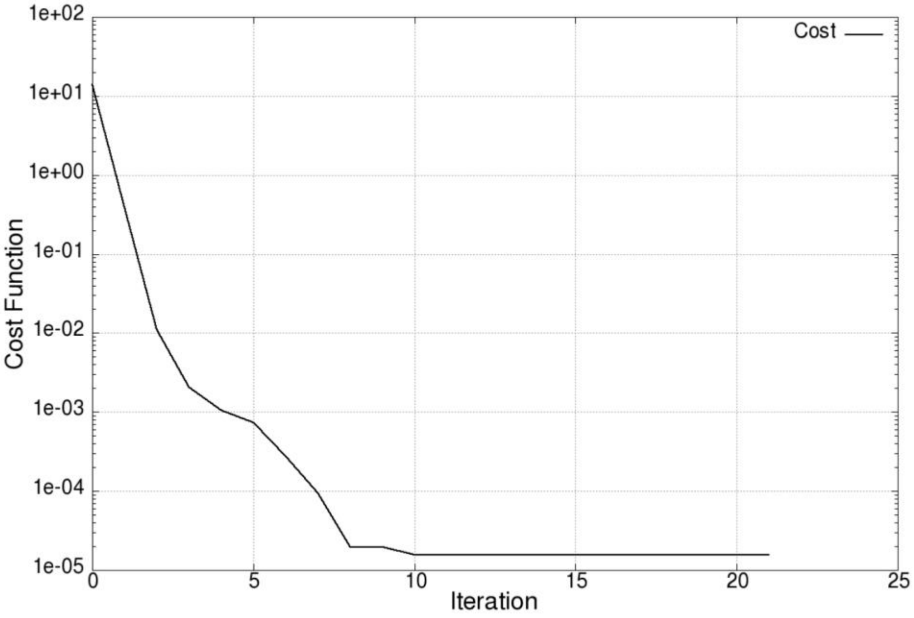
- 4 Load Cases: $F_x=(10^5, 4\cdot10^7, 5\cdot10^7, 6\cdot10^7)$
- Target:
- $E_0 = 2\cdot10^{11}$, $E_1 = 1.6\cdot10^{10}$
- $\varepsilon_0 = 2\cdot10^{-3}$, $\varepsilon_1 = 5\cdot10^{-2}$
- $\nu = 0.3$
- 4 Sensors Along the Truss



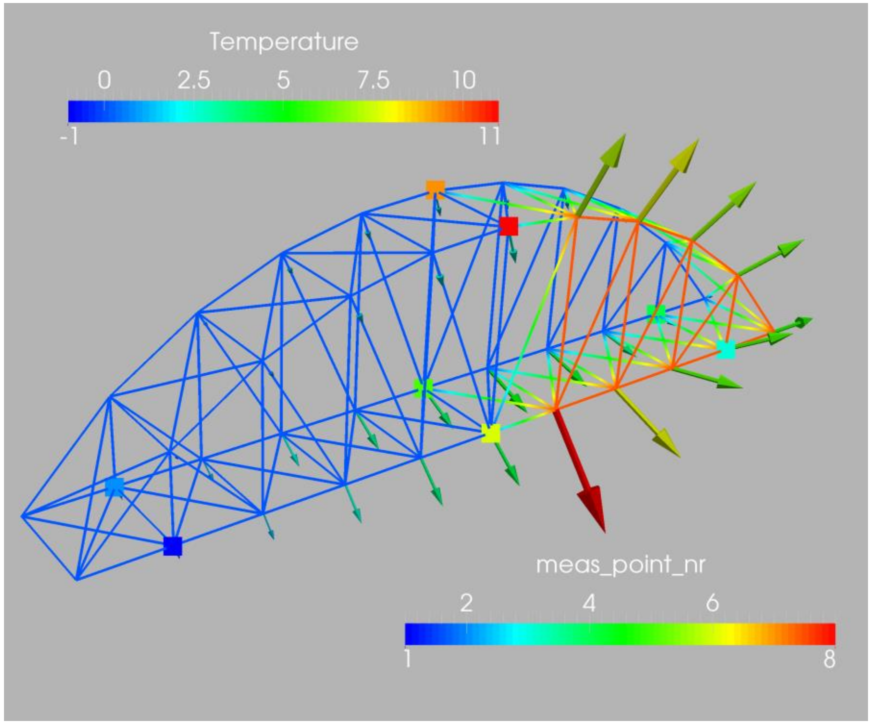
Truss (MP3)

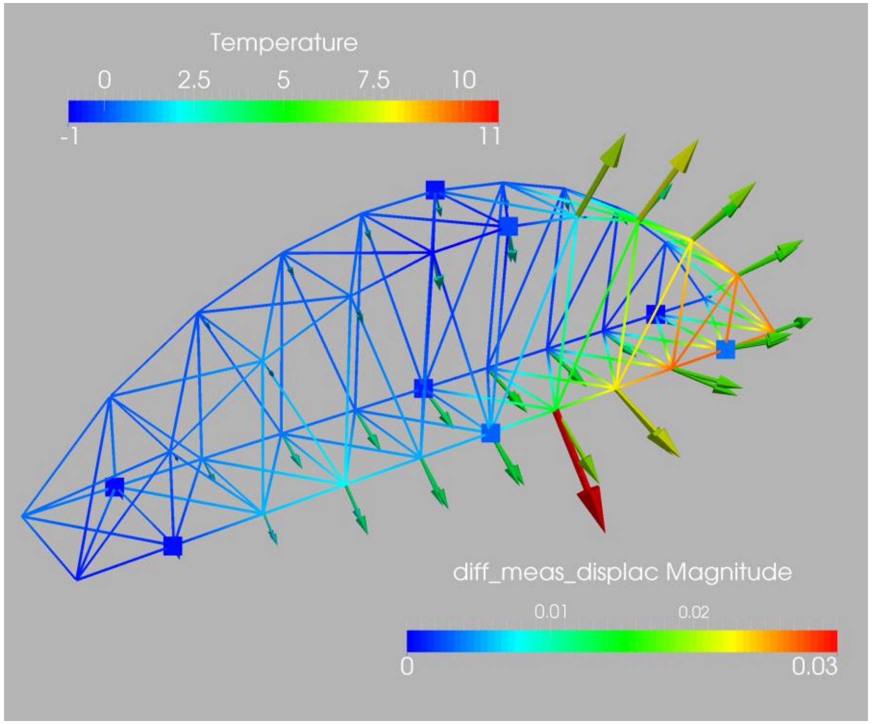
- Convergence History



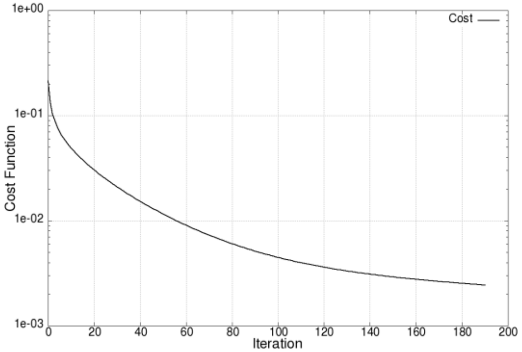


Bridge: 8 Sensors



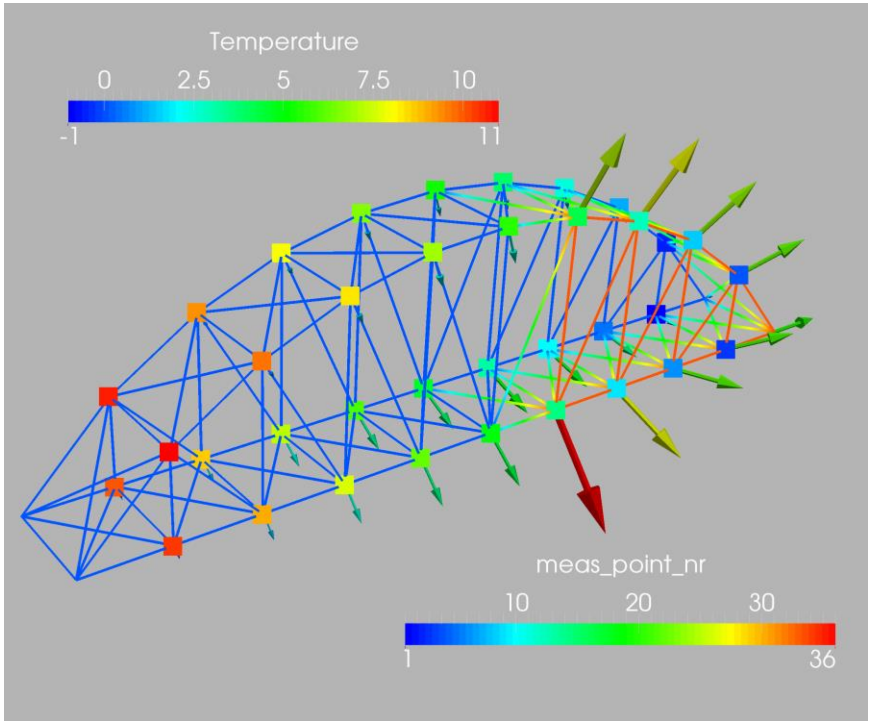


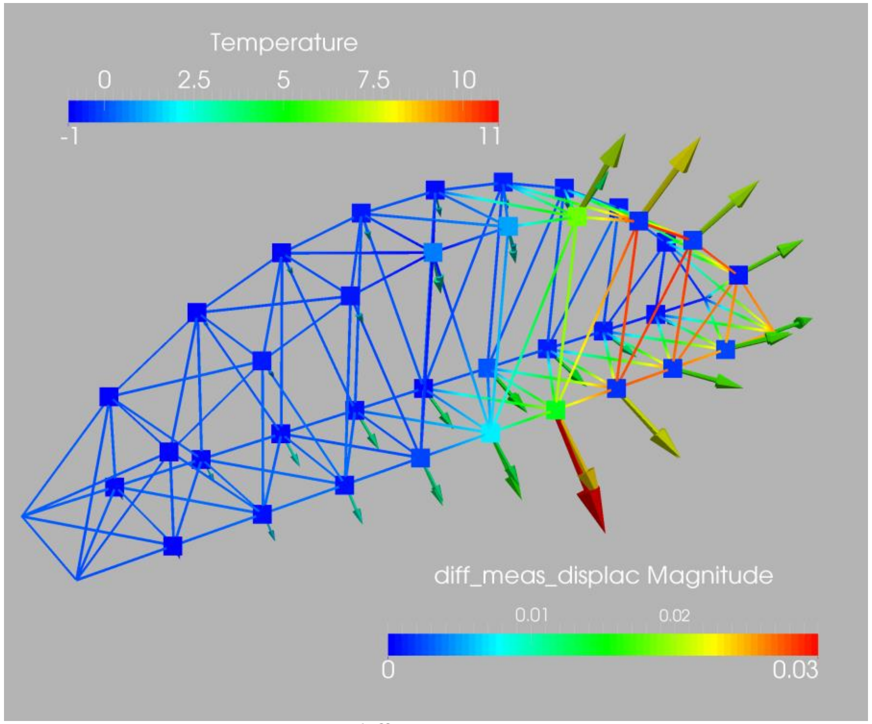
Material: Steel



FEELAST

Bridge: 36 Sensors

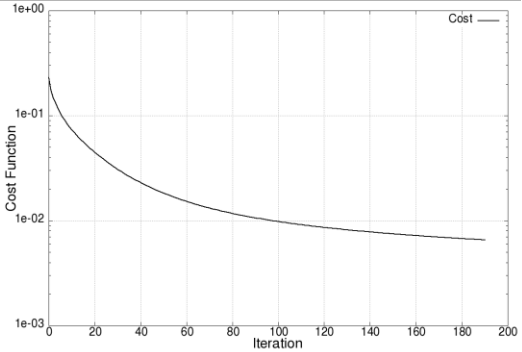




Material: Steel

Trusses: A=1-100cm^2

FEELAST



USE OF REDUCED BASIS (1)

- Detailed FEM Model Not Available/Possible

$\Rightarrow$

- Use Reduced Basis Models

Options:

- Limited (Small) Number of Eigenmodes
- Proper Orthogonal Decompositions (POCs)
- Reduced Order Methods (ROMs)

EIGENMODES (1)

Original FEM/System:

$$\mathbf{K} \cdot \mathbf{u} = \mathbf{f}$$

System of Eigenvalues λ and Eigenvectors $\mathbf{e}$:

$$\mathbf{K} \cdot \mathbf{e} = \lambda \mathbf{e}$$

$\Rightarrow$

$$\mathbf{K} \cdot \mathbf{E} = \mathbf{E} \cdot \mathbf{\Lambda}$$

- $\mathbf{E}$: Rows of Eigenvectors
- $\mathbf{\Lambda}$: Diagonal Matrix of Eigenvalues

EIGENMODES (2)

Multiplication From Right by $\mathbf{E}^{-1} = \mathbf{E}^t$

($\mathbf{K}$ Symmetric and Positive Definite):

$$\mathbf{K} = \mathbf{E} \cdot \mathbf{\Lambda} \cdot \mathbf{E}^t$$

Changing Variables and Forces via Projection:

$$\mathbf{u} = \mathbf{E} \cdot \mathbf{v} \quad , \quad \mathbf{f} = \mathbf{E} \cdot \mathbf{f}^e$$

$\Rightarrow$

$$\mathbf{\Lambda} \cdot \mathbf{v} = \mathbf{f}^e$$

$\Rightarrow$ **Uncoupled** System of Equations for Eigenmodes

Similar Decompositions Used for PODs and ROMs.

WEAKNESS DETECTION (1)

Lagrangian Functional Written With New Variables

$$L = \frac{1}{2} \sum_{i=1,m} w_i^{md} \left(\mathbf{u}^m(\mathbf{x}_i) - \mathbf{u}^{FEM}(\mathbf{x}_i) \right)^2 + \tilde{\mathbf{u}}^t \cdot (\mathbf{K} \cdot \mathbf{u} - \mathbf{f})$$

$$L = \frac{1}{2} \sum_{i=1}^{m} w_i^{md} \left(\mathbf{u}^{md}(\mathbf{x}_i) - \mathbf{E}(\mathbf{x}_i) \cdot \mathbf{v} \right)^2 + \tilde{\mathbf{v}}^t \cdot (\mathbf{\Lambda} \cdot \mathbf{v} - \mathbf{f}^e)$$

Derivatives:

$$\frac{dL}{d\tilde{\mathbf{v}}} = \mathbf{\Lambda} \cdot \mathbf{v} - \mathbf{f}^e = 0$$

$$\frac{dL}{d\mathbf{v}} = \sum_{i=1}^{m} w_i^{md} \mathbf{E}(\mathbf{x}_i) (\mathbf{u}^{md}(\mathbf{x}_i) - \mathbf{E}(\mathbf{x}_i) \cdot \mathbf{v}) + \mathbf{\Lambda} \cdot \tilde{\mathbf{v}} = 0$$

WEAKNESS DETECTION (2)

$$\frac{dL}{d\alpha_e} = \tilde{\mathbf{v}}^t \cdot \frac{d\mathbf{\Lambda}}{d\alpha_e} \cdot \mathbf{v}$$

Derivative of Each Eigenvalue λ_i WRT α_e (Appendix 1):

$$\frac{d\lambda_i}{d\alpha_e} = \frac{\mathbf{e}_i^t \cdot \mathbf{K}_e \cdot \mathbf{e}_i}{\mathbf{e}_i^t \cdot \mathbf{e}_i}$$

$\Rightarrow$ Need Stiffness Matrix of Each Element

BUT: May Not Be Available !

For Forces:

$$\frac{dL}{d\mathbf{f}_i^e} = -\tilde{\mathbf{v}}^t \cdot \frac{d\mathbf{f}^e}{d\mathbf{f}_i^e} = -\tilde{\mathbf{v}}_i^t$$

APPENDIX 1: DERIVATIVES OF EIGENVALUES (1)

Original System:

$$\mathbf{K} \cdot \mathbf{e} = \lambda \mathbf{e} \tag{$*$}$$

Change In Stiffness:

$$\Delta \mathbf{K}_e = \Delta \alpha_e \mathbf{K}_e$$

$\Rightarrow$

$$(\mathbf{K} + \Delta \mathbf{K}) \cdot (\mathbf{e} + \Delta \mathbf{e}) = (\lambda + \Delta \lambda)(\mathbf{e} + \Delta \mathbf{e})$$

Expand:

$$\mathbf{K} \cdot \mathbf{e} + \Delta \mathbf{K} \cdot \mathbf{e} + \mathbf{K} \cdot \Delta \mathbf{e} + \Delta \mathbf{K} \cdot \Delta \mathbf{e} = \lambda\, \mathbf{e} + \lambda\, \Delta \mathbf{e} + \Delta \lambda\, \mathbf{e} + \Delta \lambda\, \Delta \mathbf{e}$$

APPENDIX 1: DERIVATIVES OF EIGENVALUES (2)

Removing 0th Order Terms, Neglect 2nd Order Terms:

$$\Delta\mathbf{K} \cdot \mathbf{e} + \mathbf{K} \cdot \Delta\mathbf{e} = \lambda\ \Delta\mathbf{e} + \Delta\lambda\ \mathbf{e} \qquad (**)$$

Multiply from Left by $\mathbf{e}^t \Rightarrow$

$$\mathbf{e}^t \cdot \Delta\mathbf{K} \cdot \mathbf{e} + \mathbf{e}^t \cdot \mathbf{K} \cdot \Delta\mathbf{e} = \mathbf{e}^t \cdot \lambda \cdot \Delta\mathbf{e} + \mathbf{e}^t \cdot \Delta\lambda \cdot \mathbf{e}$$

Use Eqn.(*) Again:

$$\mathbf{e}^t \cdot \Delta\mathbf{K} \cdot \mathbf{e} = \mathbf{e}^t \cdot \Delta\lambda \cdot \mathbf{e}$$

$\Rightarrow$

$$\mathbf{e}^t \cdot \Delta\alpha_e \mathbf{K}_e \cdot \mathbf{e} = \mathbf{e}^t \cdot \Delta\lambda \cdot \mathbf{e}$$

$$\frac{\Delta\lambda}{\Delta\alpha_e} = \frac{\mathbf{e}^t \cdot \mathbf{K}_e \cdot \mathbf{e}}{\mathbf{e}^t \cdot \mathbf{e}}$$

APPENDIX 1: DERIVATIVES OF EIGENVALUES (3)

And $\frac{\Delta \mathbf{e}}{\Delta \alpha_e}$?

Using Eqn.(**):

$$(\mathbf{K} - \lambda \mathbf{I}) \cdot \Delta \mathbf{e} = (\Delta \lambda - \Delta \mathbf{K}) \cdot \mathbf{e}$$

$\Rightarrow$

$$(\mathbf{K} - \lambda \mathbf{I}) \cdot \frac{\Delta \mathbf{e}}{\Delta \alpha_e} = \left(\frac{\Delta \lambda}{\Delta \alpha_e} - \mathbf{K}_e \right) \cdot \mathbf{e}$$

Seems Costly

If Decomposition of **K** Given: Iterative Scheme

$$\mathbf{K} \cdot \left(\frac{\Delta \mathbf{e}}{\Delta \alpha_e} \right)^{i+1} = \lambda \mathbf{I} \cdot \left(\frac{\Delta \mathbf{e}}{\Delta \alpha_e} \right)^{i} + \left(\frac{\Delta \lambda}{\Delta \alpha_e} - \mathbf{K}_e \right) \cdot \mathbf{e}$$

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